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Minimal graphs for matching extensions

M.-C. Costa [a], D. de Werra [b], C. Picouleau [c,*]

[a] Ecole Nationale Supérieure des Techniques Avancées Paris-Tech and CEDRIC laboratory, Paris, France
[b] Ecole Polytechnique Fédérale de Lausanne, Lausanne, Switzerland
[c] Conservatoire National des Arts et Métiers, CEDRIC laboratory, Paris, France

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ABSTRACT

Given a positive integer n we find a graph $G = (V, E)$ on $|V| = n$ vertices with a minimum number of edges such that for any pair of non adjacent vertices x, y the graph $G - x - y$ contains a (almost) perfect matching M. Intuitively the non edge xy and M form a (almost) perfect matching of G. Similarly we determine a graph $G = (V, E)$ with a minimum number of edges such that for any matching $\bar{M}$ of the complement $\bar{G}$ of G with size $\lfloor \frac{n}{2} \rfloor - 1$, $G - V(\bar{M})$ contains an edge e. Here $\bar{M}$ and the edge e of G form a (almost) perfect matching of $\bar{G}$.

We characterize these minimal graphs for all values of n.

1. Introduction

We shall consider here a kind of reliability problem which occurs rather naturally in a context where some elements of a complex system may break down either due to attacks or simply to technical failures. We want to protect a subset of elements (as small as possible) in order to keep the system working in spite of possible failures occurring in the rest of the system.

To give a formulation in terms of graphs, we introduce definitions and notations. Given a simple finite graph $G = (V, E)$ with n vertices $v_1, v_2, \ldots, v_n$ and m edges, we denote by $\bar{G} = (V, \bar{E})$ the complement of G. For any subset $F \subseteq E$, $V(F)$ is the set of endpoints of the edges in F. For any subset $X \subseteq V$ the subgraph induced by X is denoted by $G[X]$. We write $G - X = G[V \setminus X]$ and $G - v$ for $G - \{v\}$. The union of two graphs G_1, G_2 on disjoint vertex sets without any edges between them is written $G_1 + G_2$. $N_G(v)$ is the set of neighbors of a vertex v in G; $\delta_G(v) = |N_G(v)|$ is the degree of v in G; a p-vertex is a vertex of degree p in G; if $\delta_G(v) = n - 1$ then v is *universal*. For any nonempty subset $A \subseteq V$ we denote by $N_G(A)$ the set of vertices $v \in V \setminus A$ having a neighbor in A, i.e. $N_G(A) = \bigcup_{v \in A} N_G(v) \setminus A$. Let A, B be disjoint sets of vertices. We denote by $m_G(A, B)$ the number of edges linking A and B.

A subset $M \subseteq E$ is a *matching* if no two edges in M are incident to a same vertex; $\mu(G)$ is the maximum cardinality of a matching in G. G has a *perfect* matching if $\mu(G) = n/2$ and an *almost perfect* matching if $\mu(G) = (n - 1)/2$.

For all definitions related to graphs, see [4].

We intend to determine for two given positive integers d, n a graph $G = (V, E)$ on n vertices with a minimum number of edges, such that to any matching $\bar{M}$ of d edges of $\bar{G}$ one can associate a matching of $\lfloor n/2 \rfloor - d$ edges in $G - V(\bar{M})$. Hence if the edges of $\bar{M}$ would be edges in G, then $\bar{M} \cup M$ would be a (almost) perfect matching of G. Notice that a feasible set E of

* Corresponding author.
E-mail addresses: marie-christine.costa@ensta-paristech.fr (M.-C. Costa), christophe.picouleau@cnam.fr (D. de Werra), dominique.dewerra@epfl.ch (C. Picouleau).

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edges always exists: take for instance for E the edges of a complete graph on n vertices from which we remove a matching of size d.

In our paper we determine the minimum size of *expandable* graphs G (corresponding to the case $d = 1$); these are graphs such that for any edge xy in $\bar{E}$, the subgraph $G-x-y$ has a (almost-)perfect matching. Similarly we determine the minimum size of *completable* graphs G (corresponding to the case $d = \lfloor n/2 \rfloor - 1$); these are graphs such that for any matching $\bar{M}$ of $\bar{G}$ with $|\bar{M}| = \lfloor n/2 \rfloor - 1$ there exists an edge $uv \in G - V(\bar{M})$.

In our reliability interpretation the edges of these minimal graphs G are the ones which should be protected so that one could extend the matchings $\bar{M}$ of size d to (almost)-perfect matchings in spite of failures in $\bar{G}$.

Various concepts of matching extension have been studied. Some consider these extensions in special classes of graphs [1,6,12]. In [11,12] several properties related to perfect matchings are examined. It is the case of d-extendable graphs defined as graphs in which every matching of size d can be extended to a perfect matching. In particular for $d = 1$, one requires that for any edge xy, $G-x-y$ has a perfect matching [10]. A graph is *bicritical* if for any pair $\{x, y\}$ of vertices, xy being an edge or not, $G - x - y$ has a perfect matching. Notice that the graphs considered there have a perfect matching. Clearly a bicritical graph is 1-extendable and also expandable. A claw $K_{1,3}$ is expandable but not 1-extendable and a cycle C_6 is 1-extendable but not expandable.

It is worth underlining that to our knowledge matching extensions by edges of G or $\bar{G}$ have not been associated with the optimization of the size of the graphs. This is the main motivation for this research.

In Section 2 we will characterize the expandable graphs of n vertices with a minimum number of edges. The case where the expandable graphs are constrained to be connected is treated in the third section. Then Section 4 will be devoted to completable graphs on n vertices with a minimum number of edges. Finally we will mention in the conclusion some variations and generalizations.

2. Minimal expandable graphs

We want to find a graph G with a minimum number of edges such that for every pair u, v of non adjacent vertices of G it is always possible to extend the non-edge uv to a perfect (or almost perfect) matching using only edges of G that are not incident to u or v, formally $\mu(G-u-v) = \lfloor n/2 \rfloor - 1$.

We say that G is *expandable* if for any non-edge $uv \notin E$ there exists a matching M of $G-u-v$ with $|M| = \lfloor n/2 \rfloor - 1$.

An expandable graph $G = (V, E)$ on n vertices with a minimum number of edges is a *Minimum Expandable Graph*. The size $|E|$ of its edge set is denoted by $Exp(n)$. The set of minimal expandable graphs of order n is called $MEG(n)$.

Since the problem is trivial for $n \leq 3$ we shall assume $n \geq 4$.

Proposition 2.1. *For* $4 \leq n \leq 7$ *we have:*

- $Exp(4) = 3$ *and* $MEG(4) = \{K_{1,3}, \bar{K}_{1,3}\}$;
- $Exp(5) = 3$ *and* $MEG(5) = \{K_3 + 2K_1\}$;
- $Exp(6) = 6$ *and* $MEG(6) = \{2K_3\}$;
- $Exp(7) = 6$ *and* $MEG(7) = \{2K_3 + K_1, C_5 + K_2\}$.

Proof. Let $n = 4$. One can verify that $K_{1,3}$ and its complement $\bar{K}_{1,3}$ are expandable. Suppose that there exists $G = (V, E) \in MEG(4)$ with $|E| = 2$: then G has two non adjacent 1-vertices v_1, v_2; so $\mu(G - v_1 - v_2) = 0 < 1$. The only graph with three edges non isomorphic to $K_{1,3}$ or $\bar{K}_{1,3}$ is P_4, and P_4 is not expandable.

Let $n = 5$. One can verify that $K_3 + 2K_1$ is expandable. Suppose that there exists $G = (V, E) \in MEG(5)$ with $|E| = 2$: then G has two non adjacent 1-vertices v_1, v_2; so $\mu(G - v_1 - v_2) = 0 < 1$. The only non isomorphic graphs with 3 edges are $K_3 + 2K_1$, $P_4 + K_1$, $P_3 + K_2$, $K_{1,3} + K_1$. Among them only $K_3 + 2K_1$ is expandable.

Let $n = 6$. One can verify that $2K_3$ is expandable. Suppose that there exists $G = (V, E) \in MEG(6)$ with $|E| \leq 5$: if G has a 1-vertex v_1, its neighbor v_2 must be universal otherwise $\mu(G - v_2 - v_i) < 2$, $v_i \notin N_G(v_2)$. But $G = K_{1,5}$ is clearly not expandable. So G has a 0-vertex and then the five remaining vertices must induce K_5 which has more than six edges.

We prove that the only graph in $MEG(6)$ is $2K_3$. Suppose that there exists $G \in MEG(6)$ and $G \neq 2K_3$. It cannot have a 0-vertex. If G has a 1-vertex then its neighbor must be universal and G consists of a spanning star and an additional edge; such a G is not expandable. It follows that all vertices have degree two and thus $G \in \{C_6, 2K_3\}$ but C_6 is not expandable, hence $G = 2K_3$.

Let $n = 7$. One can verify that $2K_3 + K_1$ and $C_5 + K_2$ are expandable. Suppose that there exists $G = (V, E) \in MEG(7)$ with $|E| \leq 5$: If there exists a 0-vertex u then $G - u$ must be expandable and from above $|E| \geq 6$. So there are at least four 1-vertices and two of them v_1, v_2 are in two different connected components then $\mu(G - w_1 - w_2) < 2$ where w_1, w_2 are the neighbors of v_1, v_2.

We prove that $MEG(7) = \{2K_3 + K_1, C_5 + K_2\}$. Suppose that there exist $G = (V, E) \in MEG(7)$, $|E| = 6$, and $G \neq 2K_3 + K_1$, $C_5 + K_2$. If G has one 0-vertex u then $G - u$ must be expandable: so $G - u = 2K_3$ and $G = 2K_3 + K_1$. It follows that the number k of 1-vertices in G is at least two.

Two 1-vertices cannot have a common neighbor otherwise G must be a spanning star which is clearly not expandable. Moreover, the neighbors of 1-vertices must induce a clique: if $k > 2$, since $|E| = 6$, then $k = 3$ and there is a 0-vertex: a contradiction.

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