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On the forbidden induced subgraph probe and sandwich problems

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ABSTRACT

In this work we compare the complexities of GRAPH SANDWICH PROBLEMS, PARTITIONED and UNPARTITIONED PROBE PROBLEMS. Particularly, we consider the problem of recognizing probe graphs with respect to a class of graphs defined by excluding induced subgraphs. Our main result is: for F-free graphs, where F is a 3-connected, non complete graph, we prove that if F -FREE GRAPH SANDWICH PROBLEM is NP-complete, then PARTITIONED PROBE F -FREE PROBLEM and PROBE F -FREE PROBLEM are NP-complete. We also compare partitioned and unpartitioned versions of PROBE PROBLEMS dealing with 2-connected graphs. Besides that, we consider C_k-free and $(C_4, \ldots, C_{|N|})$-free graphs and we prove that both problems in both versions of PROBE PROBLEMS are NP-complete. In contrast, we prove that PROBE $(K_r \setminus e)$-FREE is polynomially solvable, where $K_r \setminus e$ means a complete graph of size r without one edge. Finally, motivated by PROBE C_4 -FREE, we analyze the complexity of finding a stable set that touches every even cycle of size $2k$ exactly k times, which we call k -STABLE C_{2k} TRANSVERSAL. We prove that finding a k-stable C_{2k} transversal can be done in polynomial time.

1. Introduction

Interval Probe Graphs were introduced by Zhang [9] in 1994 as a new graph theoretic model and used in [10,11] to model certain problems in physical mapping of DNA. We will use a generalization of this concept (as surveyed in [2]):

Definition 1. Let $\mathcal{G}$ be a class of graphs. A graph $G = (V, E)$ is a *probe graph* $\mathcal{G}$ if its vertex set can be partitioned into a set of *probes* $P$and a stable set of *nonprobes* N such that G can be embedded in a graph of $\mathcal{G}$ by adding edges between certain nonprobes. In this case, we say that (P, N) is a $\mathcal{G}$ *probe partition* for G.

If the partition of the vertex set into *probes* P and *nonprobes* N is an input data, then we call G a *partitioned probe graph of* $\mathcal{G}$ if G can be embedded into a graph of $\mathcal{G}$ by adding some edges between nonprobe vertices. We denote a partitioned graph as $G = (P + N, E)$, and when this notation is used it is understood that we work with the partitioned problem. Moreover, we refer to UNPARTITIONED PROBE PROBLEM FOR A CLASS $\mathcal{G}$ or simply PROBE PROBLEM FOR $\mathcal{G}$, as PROBE $\mathcal{G}$. When we want to refer the partitioned version, we use the notation PP-$\mathcal{G}$.

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In 1995, as a natural generalization of RECOGNITION PROBLEMS, Golumbic, Kaplan and Shamir [8] introduced the concept of a new decision problem: GRAPH SANDWICH PROBLEMS, which can be formulated as follows:

GRAPH SANDWICH PROBLEM FOR PROPERTY Π (Π-SP)
Input: Two graphs $G^1 = (V, E^1)$ and $G^2 = (V, E^2)$ such that $E^1 \subseteq E^2$.
Question: Is there a graph $G = (V, E)$ such that $E^1 \subseteq E \subseteq E^2$ satisfying property Π?

If such a graph exists, it is called *sandwich graph*. Each edge in E^1 is a *forced edge*, while each edge of $E^2 \backslash E^1$ is called an *optional edge*. Then, every edge that is not in E^2 is considered a *forbidden edge*. We will denote by $G^3 = (V, E^3)$ the complement graph of G^2, and each edge in E^3 will be a forbidden one. We can then define GRAPH SANDWICH PROBLEMS accordingly.

GRAPH SANDWICH PROBLEM FOR PROPERTY Π (Π-SP)
Input: A triple (V, E^1, E^3), where $E^1 \cap E^3 = \emptyset$.
Question: Is there a graph $G = (V, E)$ satisfying Π such that $E^1 \subseteq E$ and $E \cap E^3 = \emptyset$?

We have a clear relation between PARTITIONED PROBE PROBLEMS and GRAPH SANDWICH PROBLEMS: the last generalizes the former. Thus, if the partitioned version of a PROBE PROBLEM for a class $\mathcal{G}$ is known to be NP-complete, so will be the GRAPH SANDWICH PROBLEM for this class. Conversely, if the GRAPH SANDWICH PROBLEM is polynomially solvable, then the PARTITIONED PROBE PROBLEM is also in P.

Intrigued with the relation between GRAPH SANDWICH PROBLEMS and PROBE PROBLEMS, we started to study how we could link known NP-complete GRAPH SANDWICH PROBLEMS and PARTITIONED PROBE PROBLEMS when we are interested in properties related to forbidden subgraphs. A graph G is *k-connected* if the removal of any set of $k-1$ vertices keeps G connected. In other words, in order to disconnect G, at least k vertices are necessary. Let F be a non complete 3-connected graph. In Section 2, we prove that if F-free-SP is NP-complete, then PP-F-FREE is NP-complete. Let F^* be a non complete 2-connected graph. We prove that if PP-F^*-FREE is NP-complete, then PROBE F^*-FREE is NP-complete as well. This result can be extended to a family $\mathcal{F}$ of 2-connected graphs.

In Section 3, we work with C_k- free graphs with fixed k and $K_4 \backslash e$-free (diamond-free graphs), which are 2-connected forbidden subgraphs. We know that C_k -FREE-SP and $(K_r \backslash e)$ -FREE-SP are respectively NP-complete and polynomially solvable [6]. We prove that, as well as the sandwich problem, PP-C_k-FREE is NP-complete and, as a corollary of results presented in Section 2, PROBE C_k-FREE is NP-complete. We also prove that PROBE $(K_r \backslash e)$-FREE is polynomial and, particularly, we observe that PROBE DIAMOND-FREE is polynomial time solvable. Finally, motivated by PROBE C_4-FREE, we began a study of stable sets that touch two times each C_4 of a given graph. This problem, which we call 2-STABLE C_4 TRANSVERSAL, could be considered the first step of a strategy to solve PROBE C_4-FREE. We prove that, although PROBE C_4-FREE is NP-complete, we can solve 2-STABLE C_4 TRANSVERSAL in polynomial time. We generalized this problem and this solution, defining k-STABLE C_{2k} TRANSVERSAL and showing that it is also a polynomial time solvable problem. We left k-STABLE C_{2k+1} TRANSVERSAL as an open problem.

2. Sandwiches, probes and forbidden induced subgraphs

In this section, we relate in a different way PROBE and SANDWICH PROBLEMS computational complexities when we are concerned with a forbidden induced subgraph property. To our knowledge, this is the first time in the literature that this relation is done. We show that, (1) If F-free-SP is NP-complete, then PP-F-FREE is NP-complete and (2) If PP-F^*-FREE is NP-complete, then PROBE F^*-FREE is NP-complete, where F and F^* are, respectively, 3-connected and 2-connected non complete graphs.

Next, we formulate these decision problems.

F-FREE GRAPH SANDWICH PROBLEM (F-FREE-SP)
Input: Two graphs $G^1 = (V, E^1)$ and $G^2 = (V, E^2)$ such that $E^1 \subseteq E^2$.
Question: Is there a F-free graph $G = (V, E)$ such that $E^1 \subseteq E \subseteq E^2$?

PARTITIONED PROBE F-FREE PROBLEM (PP-F-FREE)
Instance: A graph $G = (P + N, E)$ such that N is a stable set.
Question: Is there a F-free graph $G' = (V, E + E')$, where $E' = \{xy | x, y \in N\}$, for some vertices $x, y \in N$?

PROBE F-FREE PROBLEM (PROBE F-FREE)
Instance: A graph $G = (V, E)$.
Question: Is there a partition of V into (P, N) such that N is a stable set and that there exists a F-free graph $G' = (V, E + E')$, where $E' = \{xy | x, y \in N\}$, for some vertices $x, y \in N$?

We state Theorem 1:

Theorem 1. *If F-FREE GRAPH SANDWICH PROBLEM is NP-complete, where F is a 3-connected, non complete graph, then PARTITIONED PROBE F-FREE PROBLEM is NP-complete.*