ARTICLE IN PRESS
Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

The probabilistic minimum dominating set problem

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ARTICLE INFO

Article history:
Received 3 June 2015
Received in revised form 13 October 2016
Accepted 24 October 2016
Available online xxxx

Keywords:
Complexity
Approximation
Probabilistic optimization
Dominating set
Wireless sensor networks

ABSTRACT

We present a natural wireless sensor network problem, which we model as a probabilistic version of the MIN DOMINATING SET problem (called PROBABILISTIC MIN DOMINATING SET). We first show that calculation of the objective function of this general probabilistic problem is #**P**-complete. We then introduce a restricted version of PROBABILISTIC MIN DOMINATING SET and show that, this time, calculation of its objective function can be performed in polynomial time and that this restricted problem is "just" **NP**-hard, since it is a generalization of the classical MIN DOMINATING SET. We study the complexity of this restricted version in graphs where MIN DOMINATING SET is polynomial, mainly in trees and paths and then we give some approximation results for it.

1. Introduction: wireless sensor networks and probabilistic dominating set

Very frequently, in wireless sensor networks [37], one wishes to identify a subset of sensors, called "master" sensors, that will have a particular role in messages transmission, namely, to centralize and process messages sent by the rest of the sensors, called "slave" sensors, in the network. These latter sensors will be only nodes of intermediate messages transmission, while the former ones will be authorized to make several operations on messages received and will be, for this reason, better or fully equipped and preprogrammed.

So, the objective for designing such a network is to identify a subset of sensors (the master sensors) such that, every other sensor is linked to some sensor in this set. In other words, one wishes to find a dominating set in the graph of sensors. Since the equipment of master sensors induces some extra cost, if this cost is the same for all master sensors, we have a minimum cardinality dominating set problem (MIN DOMINATING SET), while if any master sensor has its own cost, we have a minimum weight dominating set problem.

Sensors can be broken down at any time but, since the network must always remain operational, once a sensor failure arrives, a new set of master sensors has to be recomputed very quickly (solution from scratch being very costly in time is proscribed). For simplicity, we deal with master sensors of uniform equipment cost (hopefully, it will be clear later that this assumption is not restrictive for the model) and we suppose that any sensor, can be broken down with some probability q_i (so, it remains operational, i.e., present in the network, with probability $p_i = 1 - q_i$) depending on its construction, proper equipment, age, etc.

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http://dx.doi.org/10.1016/j.dam.2016.10.016

Following the possible failures, we must be able to propose quickly a new solution that is a dominating set for the sub-network. Since nodes of the initially computed dominating set (called a *priori dominating set*) have already been equipped, we take in a first time, the track of this solution, i.e., the part of the solution in the surviving (present) sub-network. If it is feasible, then no corrective action is necessary. On the other hand, if the remaining subset is no more a dominating set, then we have to modify it with recourse (i.e., with an additional cost per elementary modification) in order to obtain a dominating set of present sub-network. To equip or program a *new* sensor has a significant cost, because of obligation to work in emergency. So, the supplementary equipments induced by the recourse will be more expensive.

In this paper we handle such a model for MIN DOMINATING SET, called PROBABILISTIC MIN DOMINATING SET. The objective of which consists of determining an *a priori* dominating set through a graph $G = (V, E)$ of n sensors, with probabilities $\Pr[V] = (p_1, \dots, p_n)$ for sensor performing well. We consider recourse models that take into account the modifications to the *a priori* dominating set in order to obtain an *a posteriori* solution, feasible for the subgraph effectively present. Without loss of generality, we assume that equipment costs for a sensor selected in the *a priori* solution are fixed to 1 and equipment costs for a sensor added in the recourse are fixed to $\alpha \geqslant 1$, for any added sensor. The goal is to minimize the total expected cost.

2. Preliminaries

The MIN DOMINATING SET problem including stochastic elements has not been very extensively studied. In [1], authors handle the connected dominating set in graphs where the weights associated to vertices are stochastic. The feasibility of the solution is not a goal there. In our context, the sensors' presence is the only stochastic element taken into account, and the difficulty comes from the fact that an initial solution (an *a priori* dominating set) does not necessarily remain feasible after the failure of some of its elements. Such a setting is analogous, for example, to the probabilistic travelling salesman problem with time windows in which each customer has a probability p_i of requiring a service on a given day and an *a priori* tour must be modified for the given day. This problem was studied in [40], following the so-called *a priori* optimization introduced by Jaillet for the probabilistic travelling salesman problem in [20,21].

Here, we also use the *a priori* optimization setting for the MIN DOMINATING SET problem. To our knowledge, it is the first time that such a recourse model is proposed for the dominating set. Here, we need to obtain a feasible *a posteriori* solution for each given subgraph and, for doing this, we need to modify the *a priori* solution. This context differs from the one of chance constrained approach for which we cannot modify this *a priori* solution and where the goal is to propose an *a priori* solution that guarantees, with a fixed probability, that the *a posteriori* solution (obtained without recourse) will be feasible. Such a model is proposed in [28,4,5] for wireless sensor networks, where sensors can fail: the aim is to assign transmission powers to the nodes of a wireless sensor network in such a way that connectivity should be guaranteed with a given level of reliability, while the total cost is minimized. Note that neither *a priori* optimization, nor chance constrained approach has ever been applied for the dominating set problem.

The framework of *probabilistic combinatorial optimization* that we adopt in this paper was introduced by [20,7]. In [2,7–10, 20–23], restricted versions of routing and network-design probabilistic minimization problems (in complete graphs) have been studied under the robustness model dealt here (called *a priori optimization*). In [3,11,12,15], the analysis of the probabilistic minimum travelling salesman problem, originally presented in [7,20], has been revisited in order to propose new efficient resolution. In [16,17], authors introduce a generalization of the probabilistic travelling salesman with time constraints and study two types of recourse. Several other combinatorial problems have been also handled in the probabilistic combinatorial optimization framework, with or without recourse, including minimum colouring [32,14], maximum independent set and minimum vertex cover [30,31], longest path [29], Steiner tree problems [35,36], minimum spanning tree [9,13].

The paper is organized as follows. In the next section, we introduce the PROBABILISTIC MIN DOMINATING SET-problem. We consider 2-stage modification strategies that in the first stage take the track of the *a priori* solution in the surviving subgraph and in the second stage, if this track is infeasible, they complete it in some way into a feasible solution. We establish the expression of the total expected cost associated to PROBABILISTIC MIN DOMINATING SET and show that, even under relatively simple and natural second stage completions (as for example greedily entering some non-dominated vertices in final solution), the calculation of its objective function is #**P**-complete. We then focus on a very simple (almost "silly") modification strategy under which PROBABILISTIC MIN DOMINATING SET is in **NP** (Section 3.3). For this version, we study in Section 4, polynomial cases, in particular when input graphs are paths, cycles or trees and we propose polynomial time algorithms for these cases. Finally, in Section 5, we give some approximation results for both the cases of identical sensor probabilities and of distinct probabilities. The main results in Section 4 imply that PROBABILISTIC MIN DOMINATING SET is polynomial in trees with degrees bounded by $O(\log n)$ and in general trees assuming identical probabilities. It remains however open if the problem is polynomial in general trees with distinct vertex-probabilities, which seems to be a difficult problem. The approximability upper bounds given in Section 5 are quite far from those known for the classical MIN DOMINATING SET problem. For instance, although MIN DOMINATING SET is approximable within ratio $O(\log n)$, PROBABILISTIC MIN DOMINATING SET is approximable within ratios $\Delta + 1$ for the recourse model and $\Delta - \ln \Delta$, for a simplified version of it where $\alpha = 1$, in the case of identical sensors, and within ratio $\Delta^2/\ln \Delta$ when heterogeneous sensors are assumed, where Δ denotes the maximum degree of the input graph. This might be due to the fact that PROBABILISTIC MIN DOMINATING SET (even in its simplified form) seems to be much harder than its deterministic counterpart.

Please cite this article in press as: N. Boria, et al., The probabilistic minimum dominating set problem, Discrete Applied Mathematics (2016), http://dx.doi.org/10.1016/j.dam.2016.10.016