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# Maximum weight relaxed cliques and Russian Doll Search revisited

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ABSTRACT

Trukhanov et al. (2013) used the Russian Doll Search (RDS) principle to effectively find maximum hereditary structures in graphs. Prominent examples of such hereditary structures are cliques and some clique relaxations intensively discussed and studied in network analysis. The effectiveness of the tailored RDS by Trukhanov et al. for *s*-plex and *s*-defective clique can be attributed to their cleverly designed incremental verification procedures used to distinguish feasible from infeasible structures. In this paper, we clarify the incompletely presented verification procedure for *s*-plex and present a new and simpler incremental verification procedure for *s*-defective cliques with a better worst-case runtime. Furthermore, we develop an incremental verification for *s*-bundle, giving rise to the first exact algorithm for solving the maximum cardinality and maximum weight *s*-bundle problems.

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## 1. Introduction

The combinatorial branch-and-bound by Östergård [8] is among the most powerful exact algorithms to identify maximum cardinality and maximum weight cliques. It follows the *Russian Doll Search* (RDS) principle originally introduced by Verfaillie et al. [11] for solving valued constraint satisfaction problems. In the context of graph theory, it is applicable to find optimal hereditary structures. In particular, Trukhanov et al. [10] solve maximum cardinality *s*-plex and *s*-defective clique problems. These are examples of relaxed cliques, which are hereditary and of interest in social network analysis (see [9,4]).

Let $G = (V, E)$ be a simple graph with finite vertex set $V$ and edge set $E$. For any subset $S \subseteq V$, the vertex-induced subgraph of $S$ is $G[S] = (S, E \cap (S \times S))$. A graph property $\Pi$ is *hereditary* on induced subgraphs if for any subset $S \subseteq V$ with $G[S]$ satisfying property $\Pi$, any subset $S' \subset S$, $S' \neq \emptyset$ induces a subgraph $G[S']$ that satisfies $\Pi$. A property $\Pi$ is *nontrivial* if it is true for all $G[S]$ induced by singleton sets $S = \{i\}$, $i \in V$ and not satisfied by every graph. A property $\Pi$ is *interesting* if there exist graphs $G$ of arbitrary size satisfying $\Pi$. Yannakakis [12] has shown that the determination of a maximum cardinality set $S$ satisfying $\Pi$, i.e., the *maximum cardinality $\Pi$ problem* is $\mathcal{NP}$-hard for $\Pi$ that are hereditary, nontrivial, and interesting. In the following, we refer to these properties as the Yannakakis properties. For given vertex weights $w_i$, $i \in V$, the *maximum weight $\Pi$ problem* seeks for a set $S$ with maximum weight $w(S) = \sum_{i \in S} w_i$ satisfying $\Pi$. For hereditary $\Pi$, the weights can be assumed to be non-negative because otherwise the corresponding vertex can never be in an optimal solution.

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One prominent example of a structure that satisfies the Yannakakis properties is the clique: A set $S \subseteq V$ is a *clique* if $G[S]$ is complete, i.e., all vertices are adjacent. Pattillo et al. [9] show that first-order clique relaxations can be derived from relaxing the distance, degree, density, or connectivity requirements of cliques. Note that cliques are perfect in the sense that they have maximum density, their vertices have maximum degree, and pairs of vertices have minimum distance and maximum connectivity in the induced subgraph. In the following, we formally introduce these graph parameters and define associated relaxed cliques that can be solved with RDS.

For $i, j \in V$, $\mathrm{dist}_G(i, j)$ is the minimum length of a path in $G$ connecting $i$ and $j$. For $s \geq 1$, $S \subseteq V$ is an $s$-clique if $\mathrm{dist}_G(i, j) \leq s$ for all $i, j \in S$. Note that $s$-cliques do not fulfill the Yannakakis properties, since they are only weakly hereditary [9, p. 14]. However, as every $s$-clique is an ordinary clique in the $s$th power graph of $G$, the search for maximum $s$-cliques can be performed with any maximum clique algorithm. Therefore, we do not consider $s$-cliques in the remainder of the paper.

Let $i \in V$ be any vertex and let $S \subseteq V$ be any subset of vertices. The set of vertices adjacent to $i$ is denoted by $N(i)$. The *vertex degree* in $G$ of vertex $i$ is $|N(i)|$ and is denoted by $\deg_G(i)$. The *minimum vertex degree* of $G$ is $\delta(G) = \min_{i \in V} \deg_G(i)$. For $s \geq 1$, $S \subseteq V$ is an *s-plex* if $\delta(G[S]) \geq |S| - s$.

The set $E(S)$ is the set of edges in $G$ with both endpoints in $S$. For $s \geq 0$, $S$ is an *s-defective clique* if $|E(S)| \geq \binom{|S|}{2} - s$.

A set $C \subset V$ is a *vertex cut* of a connected graph $G = (V, E)$ if $G[V \setminus C]$ is a disconnected graph. Note that any vertex cut $C$ has at most $|V| - 2$ elements. The vertex connectivity $\kappa(G)$ of $G$ is the size of a minimum cardinality vertex cut. For cliques $S$, $G[S]$ does not have any vertex cuts, and therefore one defines $\kappa(G[S]) = |S| - 1$. A graph is called $k$-vertex-connected if its vertex connectivity is $k$ or greater. The local connectivity $\kappa_G(i, j)$ of two different and non-adjacent vertices $i, j \in V$ is the minimum size of a vertex cut $C$ disconnecting $i$ and $j$ in $G[V \setminus C]$. For adjacent vertices $i$ and $j$, one defines $\kappa_G(i, j) = \infty$. Then, if $G$ is not a complete graph, $\kappa(G)$ equals the minimum of $\kappa_G(i, j)$ over all pairs of different vertices $i, j \in V$. Two $i$–$j$-paths are called vertex-disjoint if they have no vertices in common except $i$ and $j$. Menger's theorem [7] states that the minimum size of a vertex cut disconnecting $i$ and $j$ is equal to the maximum number of vertex-disjoint paths connecting $i$ and $j$. Hence, for non-adjacent vertices $i$ and $j$, $\kappa_G(i, j)$ is the maximum number of vertex-disjoint $i$–$j$-paths. For $s \geq 1$, $S$ is an *s-bundle* if $\kappa(G[S]) \geq |S| - s$.

Note that any (ordinary) clique $S$ is a 1-plex, 0-defective clique, and 1-bundle. For $s > 1$, every $(s-1)$-defective clique and every $s$-bundle is an $s$-plex, but the reverse is generally not true.

A prerequisite of RDS is that the $n$ vertices in $V$ are ordered into a sequence $(v_1, v_2, \dots, v_n)$. Instead of one depth-first branch-and-bound search, RDS performs $n$ searches. Starting from $i = n$, the $i$th search determines a maximum weight $\Pi$ set for $G[\{v_i, v_{i+1}, \dots, v_n\}]$ with the initial set $S = \{v_i\}$. In every iteration, $i$ is decreased by 1 so that a sequence of lower bounds $LB_n, LB_{n-1}, \dots, LB_2, LB_1$ is computed. These bounds allow an improved pruning compared to single branch-and-bound searches (see Section 2). At each stage of the RDS search, the current solution $P$ satisfies $\Pi$. Moreover, a set of candidates $C$ with $P \cup \{c\}$ satisfying $\Pi$ for all $c \in C$ is maintained. Whenever $P$ is enlarged, $C$ has to be adjusted, i.e., candidate vertices not compatible with the new set $P$ are removed from $C$. The test whether $P \cup \{c\}$ for a candidate vertex $c \in C$ satisfies $\Pi$ is called the *verification procedure*.

Trukhanov et al. [10] presented straightforward and incremental verification procedures for $s$-plex and $s$-defective clique. While straightforward procedures are simpler to implement, the incremental verification procedures have a better runtime complexity.

The contribution of this paper is threefold: First, we clarify the incremental verification procedure for $s$-plex because the description in [10] is incomplete. Second, we present a new and simpler incremental verification procedure for $s$-defective cliques with a better worst-case runtime complexity. Third, no solution algorithm for $s$-bundle neither heuristic nor exact has been presented in the literature. We develop an incremental verification procedure and herewith introduce a first, RDS-based algorithm for maximum-weight $s$-bundle.

The remainder of the paper is structured as follows: In Section 2, we briefly summarize RDS and present the new incremental verification procedures. In Section 3, the effectiveness of the new RDS algorithms is analyzed in a computational study. Final conclusions are drawn in Section 4.

## 2. Russian doll search

Algorithm 1 presents RDS for the maximum weight $\Pi$ problem in a slightly modified version compared to [10]. Different strategies for the vertex ordering in Step 1 were discussed and analyzed by Trukhanov et al. [10]. For unit weights, they state that a degree based ordering as suggested by Carraghan and Pardalos [3] turned out to give the best overall performance for their RDS. Herein, $v_n$ is first chosen as a minimum degree vertex. Then, iteratively from $i = n - 1$ down to 1, the vertex $v_i$ is selected such that $v_i$ has minimum degree in $G[V \setminus \{v_{i+1}, \dots, v_n\}]$.

RDS maintains $n+1$ lower bounds: The global bound $LB$ (Step 2) is the weight of the best solution found so far. Moreover, the $n$ branch-and-bound searches are initiated in the main loop (Steps 3–6). Each search produces a best solution of weight $LB_i$ by calling the procedure FindMax which performs the actual branch-and-bound on $G[\{v_i, \dots, v_n\}]$. The initial candidate set $C$ is computed in Step 4 using a problem-specific verification procedure.

The Procedure FindMax is called with the candidate set $C$ and the current set $P$. RDS always keeps $C$ and $P$ such that $P \cup \{c\}$ satisfies $\Pi$ for each $c \in C$ (in the following referred to as *consistency*). Therefore, if $C$ is empty (Steps 1–4), an inclusion maximal solution is found and tested for optimality.

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