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# Liar's dominating sets in graphs

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ABSTRACT

A set $L \subseteq V$ of a graph $G = (V, E)$ is a liar's dominating set if (1) for every vertex $u \in V$, $|N[u] \cap L| \geq 2$ and (2) for every pair $u, v \in V$ of distinct vertices, $|(N[u] \cup N[v]) \cap L| \geq 3$. In this paper, we first provide a characterization of graphs $G$ with $\gamma_{LR}(G) = |V|$ as well as the trees $T$ with $\gamma_{LR}(T) = |V| - 1$. Then we present some bounds on the liar's domination number, especially an upper bound for the ratio between the liar's domination number and the double domination number is established for connected graphs with girth at least five. Finally, we determine the exact value of the liar's domination number for the complete $r$-partite graphs.



## 1. Introduction

We consider finite, undirected, and simple graphs $G$ with vertex set $V = V(G)$ and edge set $E = E(G)$. The number of vertices $|V(G)|$ of a graph $G$ is called the *order* of $G$ and is denote by $n$. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = N_G(v) = \{u \in V \mid uv \in E\}$, and its *closed neighborhood* is the set $N[v] = N_G[v] = N(v) \cup \{v\}$. The *degree* of $v$ is the cardinality of its open neighborhood. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*.

A subset $S \subseteq V$ is a *dominating set* of $G$ if every vertex in $V \backslash S$ has a neighbor in $S$, that is, $|N[v] \cap S| \geq 1$ for all $v \in V$. A set $D \subseteq V$ is a double (2-tuple) dominating set of $G$ if $|N[v] \cap D| \geq 2$ for all $v \in V(G)$. The *double domination number*, denoted $\gamma_{\times 2}(G)$, is the minimum cardinality of a double dominating set of $G$. Domination in graphs and its variations have been extensively studied over the past three decades since they are used to model many practical problems of operations research. For further details, the reader is referred to the books of Haynes et al., [4] and [5], which provide an overview of the results before the year 1998.

In this paper, we are interested in a new variant of domination, namely liar's domination introduced by Slater in [12] as follows: a graph could be used for many structures (like a computer network, a telecommunication or sensor network, a building or a railroad network) where each vertex denotes some location in any network. In each network's location there could appear some intruder event, such as a thief, a saboteur or a fire whose location needs to be well detected and identified. A protection device placed at a vertex $v$ is assumed to be able to detect the intruder at any vertex in $N[v]$ and to specify at which vertex $u \in N[v]$ the intruder is located. However, although all protective devices detect the intruder location but they might misreport through a transmission error about the location of the intruder. A *liar's dominating set* of a graph $G = (V, E)$ is defined in [12] to be a set $D \subseteq V$ such that if for any designated vertex $x \in V$ (namely, the intruder location) if all or all but one of the vertices in $N[x] \cap D$ report vertex $x$, and at most one vertex $w$ in $N[x] \cap D$ either reports a vertex $y \in N[w] \backslash \{x\}$ or

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fails to report any vertex, then the vertex $x$ can be correctly identified as the designated vertex. The minimum cardinality of a liar's dominating set is called the *liar's domination number* and is denoted by $\gamma_{LR}(G)$ of $G$. A liar's dominating set of $G$ with cardinality $\gamma_{LR}(G)$ is called a $\gamma_{LR}(G)$*-set*. A vertex $x$ of $G$ is said to be *liar's dominated* by $L \subseteq V$ if $L$ can correctly identify $x$ as a designated vertex.

It is worth mentioning that Slater [12] was the first to show that determining the liar's domination number is NP-Complete for arbitrary graphs. Thereafter, Roden and Slater [11] showed that the problem remains NP-Complete even when restricted to bipartite graphs. It is therefore natural to look for good upper and lower bounds on the liar's domination number, and also to characterize extremal graphs for which these bounds are attained. For further details and results on liar's domination, see for examples [9–13].

We start with the following result due to Slater [12] providing a characterization of a liar's dominating set of a graph $G$.

**Theorem 1** (*Slater [12]*)**.** *A set $L \subseteq V$ of a graph $G = (V, E)$ is a liar's dominating set if and only if (1) $L$ double dominates every vertex of $V$ and (2) for every pair $u, v$ of distinct vertices we have $|(N[u] \cup N[v]) \cap L| \geq 3$.*

By Theorem 1, a graph $G$ has a liar's dominating set if and only if each component of $G$ has order at least three. Throughout this paper, we consider only such graphs. A connected subgraph $B$ of $G$ is a *block* if $B$ has no cut vertex and every subgraph $B' \subseteq G$ with $B \subseteq B'$ and $B \neq B'$ has at least one cut vertex. A block $B$ of $G$ is called an *end-block* if $B$ contains at most one cut vertex of $G$.

The rest of this paper is organized as follows. In Section 2, we provide a characterization of graphs $G$ with $\gamma_{LR}(G) = |V(G)|$ as well as a characterization of trees $T$ with $\gamma_{LR}(T) = |V(T)| - 1$. In Section 3, we present some bounds on the liar's domination number. In particular we establish an upper bound for the ratio $\gamma_{LR}(G)/\gamma_{\times 2}(G)$ for all connected graphs with girth at least five. Finally, we determine the exact value of the liar's domination number for the complete $r$-partite graphs.

## 2. Graphs with large liar's domination number

Our aim in this section is first to characterize all connected graphs $G$ of order $n$ for which $\gamma_{LR}(G) = n$, and subsequently to provide a characterization of trees $T$ of order $n$ such that $\gamma_{LR}(T) = n - 1$. It is worth pointing out that Slater [12] provided a characterization of trees $T$ of order $n$ with $\gamma_{LR}(T) = n$. Let $\mathcal{H}$ be the family of all trees $T$ such that for each vertex $v$ of $T$, $v$ is a leaf or at least one component of $T - v$ has cardinality at most two.

**Theorem 2** (*Slater [12]*)**.** *For a tree $T$ of order $n$, $\gamma_{LR}(T) = n$ if and only if $T \in \mathcal{H}$.*

We need the following definition.

**Definition 3.** Let $\mathcal{F}$ be the family of all the graphs $G$ in which each vertex $v \in V$ satisfies one of the following conditions:

(i) $v$ is a leaf,
(ii) at least one component of $G - v$ has order at most two,
(iii) $v$ belongs to an end-block of order 3.

**Theorem 4.** *For a connected graph $G$ of order $n \geq 3$, $\gamma_{LR}(G) = n$ if and only if $G \in \mathcal{F}$.*

**Proof.** Let $G$ be a graph such that $\gamma_{LR}(G) = n$, and let $v$ be any vertex of $V$. Thus $L = V \backslash \{v\}$ is not a liar's dominating set of $G$. Hence Condition (1) or (2) of Theorem 1 is not fulfilled. Suppose first that Condition (1) is not fulfilled, and let $w$ be a vertex of $V$ not double dominated by $L$. If $w = v$, then $v$ is a leaf and so (i) holds. If $w \neq v$, then $w$ has no neighbor in $L$, implying that $G - v$ has a component of order one. Hence (ii) holds. From now on, we may assume that Condition (1) holds for every vertex of $G$. Since $L$ is not a liar's dominating set of $G$, Condition (2) is not satisfied. We can assume that $v$ has degree at least two. Let $x, y$ be two distinct vertices of $G$ such that $|(N[x] \cup N[y]) \cap L| < 3$. Since Condition (1) holds for $x$, $|(N[x] \cup N[y]) \cap L| \geq 2$, implying that $|(N[x] \cup N[y]) \cap L| = 2$. Hence $xy \in E$. If $v \notin \{x, y\}$, then $\{x, y\}$ induces a component of order at most two in $G - v$, and so (ii) holds. Thus, without loss of generality, we assume that $v = x$, and let $z$ be the second vertex of $(N[x] \cup N[y]) \cap L$. Clearly $zv \in E$ and so $N(v) \cap L = \{y, z\}$. Since Condition (1) is fulfilled, $yz \in E$, for otherwise $y$ is an isolated vertex in $L$. It follows that $z$ is the unique neighbor of $y$ in $L$, and therefore $\{v, y, z\}$ induces a block of order three, in which each of $v$ and $y$ has degree two. Clearly such a block is an end-block, and hence (iii) holds.

Conversely, let $G \in \mathcal{F}$. For a contradiction, we assume that $L$ is a liar's dominating set of $G$ of cardinality less than $n$. Thus each component of the subgraph induced by $L$ has order at least three. Since every vertex $v$ not in $L$ is double dominated by $L$, $v$ does not satisfy Conditions (i) and (ii). Moreover, since every pair of distinct vertices satisfy (2) of Theorem 1, $v$ cannot belong to an end-block of order three. It follows that $v$ does not satisfy Condition (iii), which contradicts the fact that $G \in \mathcal{F}$. We conclude that $\gamma_{LR}(G) = n$. □

For the purpose of characterizing trees $T$ of order $n$ with $\gamma_{LR}(T) = n - 1$, we introduce some definitions. For a tree $T_i \in \mathcal{H}$, let $A(T_i)$ be the set of vertices that are neither leaves nor support vertices. A vertex $v \in A(T_i)$ is said to be *weak* if $T_i - v$ contains exactly one component of size two. Hence if $v$ is a weak vertex of a tree $T_i$, then it is adjacent to a support vertex of degree two. Let us call such a support vertex a *special support vertex*. Let $T_i$ be a tree of order at least four and $y$ a vertex