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On the intersection of tolerance and cocomparability graphs[☆]

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ABSTRACT

Tolerance graphs have been extensively studied since their introduction, due to their interesting structure and their numerous applications, as they generalize both interval and permutation graphs in a natural way. It has been conjectured by Golumbic, Monma, and Trotter in 1984 that the intersection of tolerance and cocomparability graphs coincides with bounded tolerance graphs. Since cocomparability graphs can be efficiently recognized, a positive answer to this conjecture in the general case would enable us to efficiently distinguish between tolerance and bounded tolerance graphs, although it is NP-complete to recognize each of these classes of graphs separately. This longstanding conjecture has been proved under some – rather strong – *structural* assumptions on the input graph; in particular, it has been proved for complements of trees, and later extended to complements of bipartite graphs, and these are the only known results so far. Furthermore, it is known that the intersection of tolerance and cocomparability graphs is contained in the class of trapezoid graphs. Our main result in this article is that the above conjecture is true for every graph G that admits a tolerance representation with exactly one unbounded vertex; note that this assumption concerns only the given tolerance *representation* R of G, rather than *any* structural property of G. Moreover, our results imply as a corollary that the conjecture of Golumbic, Monma, and Trotter is true for every graph $G = (V, E)$ that has no three independent vertices $a, b, c \in V$ such that $N(a) \subset N(b) \subset N(c)$, where $N(v)$ denotes the set of neighbors of a vertex $v \in V$; this is satisfied in particular when G is the complement of a triangle-free graph (which also implies the above-mentioned correctness for complements of bipartite graphs). Our proofs are constructive, in the sense that, given a tolerance representation R of a graph G, we transform R into a bounded tolerance representation R^* of G. Furthermore, we conjecture that any *minimal* tolerance graph G that is not a bounded tolerance graph, has a tolerance representation with exactly one unbounded vertex. Our results imply the non-trivial result that, in order to prove the conjecture of Golumbic, Monma, and Trotter, it suffices to prove our conjecture.

1. Introduction

A simple undirected graph $G = (V, E)$ on n vertices is called a *tolerance* graph if there exists a collection $I = \{I_u \mid u \in V\}$ of closed intervals on the real line and a set $t = \{t_u \mid u \in V\}$ of positive numbers, such that for any two vertices $u, v \in V$, $uv \in E$ if and only if $|I_u \cap I_v| \geq \min\{t_u, t_v\}$. The pair $\langle I, t\rangle$ is called a *tolerance representation* of G. A vertex u of G is called a

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bounded vertex (in a certain tolerance representation $\langle I, t\rangle$ of G) if $t_u \leq |I_u|$; otherwise, u is called an *unbounded vertex* of G. If G has a tolerance representation $\langle I, t\rangle$ where all vertices are bounded, then G is called a *bounded tolerance* graph and $\langle I, t\rangle$ a *bounded tolerance representation* of G.

Tolerance graphs find numerous applications in constrained-based temporal reasoning, data transmission through networks to efficiently scheduling aircraft and crews, as well as contributing to genetic analysis and studies of the brain [12,13]. This class of graphs has been introduced in 1982 [10] in order to generalize some of the well known applications of interval graphs. The main motivation was in the context of resource allocation and scheduling problems, in which resources, such as rooms and vehicles, can tolerate sharing among users [13]. Since then, tolerance graphs have attracted many research efforts [2,4,8,11–14,16,18–20], as they generalize in a natural way both interval graphs (when all tolerances are equal) and permutation graphs [10] (when $t_i = |I_i|$ for every $i = 1, 2, \ldots, n$); see [13] for a detailed survey.

Given an undirected graph $G = (V, E)$ and a vertex subset $M \subseteq V$, M is called a *module* in G, if for every $u, v \in M$ and every $x \in V \setminus M$, x is either adjacent in G to both u and v or to none of them. Note that $\emptyset$, V, and all singletons $\{v\}$, where $v \in V$, are trivial modules in G. A *comparability* graph is a graph which can be transitively oriented. A *cocomparability* graph is a graph whose complement is a comparability graph. A *trapezoid* (resp. *parallelogram* and *permutation*) graph is the intersection graph of trapezoids (resp. parallelograms and line segments) between two parallel lines L_1 and L_2 [9]. Such a representation with trapezoids (resp. parallelograms and line segments) is called a *trapezoid* (resp. *parallelogram* and *permutation*) *representation* of this graph. A graph is bounded tolerance if and only if it is a parallelogram graph [2]. The class of permutation graphs is a strict subset of the class of parallelogram graphs [3]. Furthermore, the class of parallelogram graphs is a strict subset of the class of trapezoid graphs [23], and both classes are subsets of the class of cocomparability graphs [9,13]. On the other hand, not every tolerance graph is a cocomparability graph [9,13].

Cocomparability graphs have received considerable attention in the literature, mainly due to their interesting structure that leads to efficient algorithms for several NP-hard problems, see e.g. [5,6,13,17]. Furthermore, the intersection of the class of cocomparability graphs with other graph classes has interesting properties and coincides with other widely known graph classes. For instance, the intersection of the class of cocomparability graphs with the class of chordal graphs is the class of interval graphs [9], while its intersection with the class of comparability graphs is the class of permutation graphs [9,22]. These structural characterizations produce direct algorithmic implications for the recognition problem of interval and permutation graphs, respectively, since the class of cocomparability graphs can be recognized efficiently [9,24]. In this context, the following conjecture has been made in 1984 [11]:

Conjecture 1 (*[11]*). *The intersection of the class of cocomparability graphs with the class of tolerance graphs is exactly the class of bounded tolerance graphs.*

Note that the inclusion in one direction is immediate: every bounded tolerance graph is a cocomparability graph [9,13], as well as a tolerance graph by definition. Conjecture 1 is a longstanding open question (cf. the open problems section of [13]); it has been proved for complements of trees [1], and later extended to complements of bipartite graphs [21], and these are the only known results so far. Furthermore, it has been proved that the intersection of the classes of tolerance and cocomparability graphs is contained in the class of trapezoid graphs [8]. Since cocomparability graphs can be efficiently recognized [24], a positive answer to Conjecture 1 would enable us to efficiently distinguish between tolerance and bounded tolerance graphs, although it is NP-complete to recognize each of these classes of graphs separately [19]. Only little is known so far about the separation of tolerance and bounded tolerance graphs; a recent work can be found in [7]. An intersection model for general tolerance graphs has been recently presented in [18], given by 3-dimensional parallelepipeds. For a brief description of this intersection model we refer to Section 2 (see also Fig. 1(a) and (b) for an illustration). This *parallelepiped representation* of tolerance graphs generalizes the parallelogram representation of bounded tolerance graphs; the main idea is to exploit the third dimension to capture the information given by unbounded tolerances. Furthermore, this model proved to be a powerful tool for designing efficient algorithms for general tolerance graphs [18].

Our contribution. Our main result is that Conjecture 1 is true for every graph G, for which there exists a tolerance representation with exactly one unbounded vertex. Furthermore, we state a new conjecture (cf. Conjecture 2 below) regarding the *minimal* separating examples between tolerance and bounded tolerance graphs. Unlike Conjecture 1, our conjecture does not concern any other class of graphs, such as cocomparability or trapezoid graphs. In order to state Conjecture 2, we first define a graph G to be a *minimally unbounded tolerance* graph, if G is tolerance but not bounded tolerance, while G becomes a bounded tolerance graph if we remove an arbitrary vertex of G.

Conjecture 2. *Any minimally unbounded tolerance graph has a tolerance representation with exactly one unbounded vertex.*

Our results imply the non-trivial result that, in order to prove Conjecture 1, it suffices to prove Conjecture 2. To the best of our knowledge, Conjecture 2 is true for all known examples of minimally unbounded tolerance graphs in the literature (see e.g. [13]).

All our results are based (a) on the 3-dimensional parallelepiped representation of tolerance graphs [18] and (b) on the fact that every graph G that is both a tolerance and a cocomparability graph, has a trapezoid representation R_T [8]. Specifically, in order to prove our results, we define three conditions on the unbounded vertices of G (in the parallelepiped representation R of G). Condition 1 states that R has exactly one unbounded vertex. Condition 2 states that, for every unbounded vertex u of G (in R), there exists no unbounded vertex v whose neighborhood is strictly included in the neighborhood of u.