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On the complexity of the shortest-path broadcast problem

Pierluigi Crescenzi [a], Pierre Fraigniaud [b,*], Magnus Halldorsson [c], Hovhannes A. Harutyunyan [d], Chiara Pierucci [a], Andrea Pietracaprina [e], Geppino Pucci [e]

[a] University of Florence, Italy
[b] CNRS and University Paris Diderot, France
[c] Reykjavik University, Iceland
[d] Concordia University, Canada
[e] University of Padova, Italy

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ABSTRACT

We study the *shortest-path broadcast* problem in graphs and digraphs, where a message has to be transmitted from a source node s to all the nodes along shortest paths, in the classical *telephone* model. For both graphs and digraphs, we show that the problem is equivalent to the broadcast problem in *layered directed* graphs. We then prove that this latter problem is NP-hard, and therefore that the shortest-path broadcast problem is NP-hard in graphs as well as in digraphs. Nevertheless, we prove that a simple polynomial-time algorithm, called MDST-broadcast, based on min-degree spanning trees, approximates the optimal broadcast time within a multiplicative factor $\frac{3}{2}$ in 3-layer digraphs, and $O(\frac{\log n}{\log\log n})$ in arbitrary multi-layer digraphs. As a consequence, one can approximate the optimal shortest-path broadcast time in polynomial time within a multiplicative factor $\frac{3}{2}$ whenever the source has eccentricity at most 2, and within a multiplicative factor $O(\frac{\log n}{\log\log n})$ in the general case, for both graphs and digraphs. The analysis of MDST-broadcast is tight, as we prove that this algorithm cannot approximate the optimal broadcast time within a factor smaller than $\Omega(\frac{\log n}{\log\log n})$.

1. Introduction

1.1. The general context

Broadcasting refers to the task in which one message has to be transmitted from one source node to all the other nodes in a network (we always assume that all nodes are reachable from the source). Constructing efficient broadcast protocols, that is, computing an appropriate scheduling for the communications between nodes, has been the source of a huge amount of work whose nature depends highly on the communication model. In this paper, we use the classical *telephone model* [15]. In this model, the network is modeled as a connected undirected or directed graph,[1] and communications proceed in a sequence of synchronous rounds. At each round, every node which is aware of the message (that is, either the source, or a

* Corresponding author. Tel.: +33 157 27 92 60; fax: +33 157 27 94 09.
E-mail address: pierre.fraigniaud@liafa.univ-paris-diderot.fr (P. Fraigniaud).

1 In this paper, "directed graph" is abbreviated to "digraph", while "undirected graph" is abbreviated to "graph".

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node that has received the message during some previous round) can forward the message to at most one of its neighbors in the network. In a digraph, the message can only travel in the direction of the edge along which it is sent. The measure of complexity is the number of rounds necessary to complete broadcast. Given a graph or a digraph $G = (V, E)$, and a node $s \in V$, we denote by $b(G, s)$ the minimum number of rounds required to broadcast a message from s to all nodes in V in the telephone model.

Given a (di)graph $G = (V, E)$ and $s \in V$, the *telephone broadcast problem* consists in computing $b(G, s)$. In the *multicast* version of the problem, a set S of terminals is additionally specified, and the objective is to compute the minimum number of rounds to inform all nodes in S (the message can of course be relayed by non-terminal nodes). In fact, in both variants of the problem, we are also interested in computing the optimal communication schedule enabling to reach the optimal broadcast or multicast time. Since no nodes need to be informed twice, this schedule can be represented by a tree T rooted at the source, spanning the terminals, with downward edges from each node u labeled by pairwise distinct integers in $[1, \deg(u)]$ where $\deg(u)$ is the number of children of u in T, specifying the order in which u's children should be informed. (Observe that w.l.o.g., we are restricting our attention to schedules where transmissions from a node occur at consecutive rounds.)

The broadcast time of many classical networks is known (cf., e.g., [10,13,15–17] and the references therein), and several efficient randomized broadcast protocols have been proposed [12]. However, the broadcast problem (and thus the multicast problem as well) is known to be NP-complete in graphs (and thus in digraphs as well) [11]. In fact, it is even known that it is NP-hard to approximate the broadcast time within a ratio $3 - \epsilon$ for any $\epsilon > 0$ [6]. There have been several attempts to design polynomial-time approximation algorithms for the broadcast and multicast problems [2,6,7,18,21], and the best known approximation ratio is $O(\frac{\log k}{\log\log k})$ for k-terminal multicast (in the case of undirected graphs), due to [7]. In directed graphs, the broadcast problem appears to be even more difficult to approximate: not only is it unlikely that there exists a polynomial-time approximation scheme for it, but it is even unlikely that it is in APX. Indeed, it has been proved that, unless $\mathrm{NP} \subseteq \mathrm{DTIME}(n^{O(\log n)})$, the broadcast problem in digraphs cannot be approximated within a ratio less than $\Omega(\sqrt{\log n})$ [6]. The best known approximation algorithm for broadcast in digraphs has approximation factor $O(\log n)$ [6]. The difficulty appears to be even more severe regarding multicast, for which it is known [9] that the k-terminal multicast time cannot be approximated within a factor less than $\Omega(\log k)$. The best polynomial-time algorithm known approximates multicast within a multiplicative factor of $O(\log k)$, but with an additive factor of $O(\sqrt{k})$ [8].

1.2. Our results

In this paper, we are interested in the *shortest-path broadcast* problem in graphs and digraphs [14] (see also [3]). Shortest path broadcast refers to the broadcast problem in which the message must reach every node u along a shortest path from the given source s to u in the given (di)graph G. In other words, the message can only traverse edges of the layered digraphs induced by the edges of the original (di)graph from a node at distance i from s to a node at distance $i + 1$ from s in G.

We first show that the shortest-path broadcast problem in graphs and digraphs is equivalent to the broadcast problem in *layered* directed graphs (a.k.a. *multi-stage* digraphs). Using this equivalence, we then show that the shortest-path broadcast problem is NP-hard in graphs and digraphs. Nevertheless, using the techniques in [21], we prove that an approximation algorithm based on a minimum-degree spanning tree construction, has approximation ratio $O(\frac{\log n}{\log\log n})$ for the shortest-path broadcast problem in general graphs and digraphs. The bad news is that this bound is tight for this algorithm, as we prove that it cannot provide an approximation ratio better than $\Omega(\frac{\log n}{\log\log n})$ for the problem. Finally, for the instances in which the source has eccentricity 2, we show that shortest-path broadcast time can be approximated within a multiplicative factor $\frac{3}{2}$ for both graphs and digraphs.

1.3. Structure of the paper

We provide the formal definition of our problems in Section 2, where we also establish the equivalence between the shortest-path broadcast problems and the broadcast problem in layered digraphs. In Section 3, we prove that all our problems are NP-hard. Section 4 is then dedicated to the design and analysis of the approximation algorithm based on the minimum-degree spanning tree construction, while Section 5 analyzes this algorithm for the instances in which the source has bounded eccentricity. We conclude by some considerations about multicast, and the potential fixed parameter tractability nature of the shortest-path broadcast problem where the parameter is the eccentricity of the source, including some open questions.

2. Definitions and preliminary results

In this paper, we focus on shortest-path broadcast, that is, a variant of the classical broadcast problem, where the message is restricted to travel along shortest paths. In other words, the message can only be transferred from a node at distance i from the source s to a node at distance $i + 1$ from s, for some $i \geq 0$. As we mentioned in the introduction, shortest-path broadcast is closely related to broadcast in layered graphs. In this section, we formalize this statement. For that purpose, let us define formally the shortest-path broadcast problems we are interested in.

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