Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Complexity of total outer-connected domination problem in graphs

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ARTICLE INFO

Article history:
Received 6 March 2014
Received in revised form 28 April 2015
Accepted 4 May 2015
Available online xxxx

Keywords:
Domination
Total outer-connected domination
Graph algorithms
Chordal graph
NP-complete
APX-complete

ABSTRACT

A set $D \subseteq V$ is called a total dominating set of a graph $G = (V, E)$, if for all $v \in V$, $|N_G(v) \cap D| \geq 1$. A total dominating set D of $G = (V, E)$ is called a total outer-connected dominating set if $G[V \setminus D]$ is connected. The Minimum Total Outer-connected Domination problem for a graph $G = (V, E)$ is to find a total outer-connected dominating set of minimum cardinality. The Total Outer-connected Domination Decision problem, the decision version of the Minimum Total Outer-connected Domination problem, is known to be NP-complete for general graphs. In this paper, we strengthen this NP-completeness result by showing that the Total Outer-connected Domination Decision problem remains NP-complete for chordal graphs, split graphs, and doubly chordal graphs. We prove that the Total Outer-connected Domination Decision problem can be solved in linear time for bounded tree-width graphs. We, then, propose a linear time algorithm for computing the total outer-connected domination number, the cardinality of a minimum total outer-connected dominating set, of a tree. We prove that the Minimum Total Outer-connected Domination problem cannot be approximated within a factor of $(1 - \varepsilon) \ln |V|$ for any $\varepsilon > 0$, unless NP $\subseteq$ DTIME $(|V|^{O(\log \log |V|)})$. Finally, we show that the Minimum Total Outer-connected Domination problem is APX-complete for bounded degree graphs.

1. Introduction

Let $G = (V, E)$ be a graph. For $v \in V(G)$, the sets $N_G(v) = \{u \in V(G) | uv \in E\}$ and $N_G[v] = N_G(v) \cup \{v\}$ denote the *open neighborhood* and the *closed neighborhood* of v in G, respectively. For a set $S \subseteq V$, the sets $N_G(S) = \cup_{u \in S} N_G(u)$ and $N_G[S] = N_G(S) \cup S$ denote the *open neighborhood* and the *closed neighborhood* of S, respectively. Let S_1 and S_2 be subsets of V. We say S_1 *dominates* S_2 if $S_2 \subseteq N_G[S_1]$, while S_1 *totally dominates* S_2 if $S_2 \subseteq N_G(S_1)$. For a set $S \subseteq V$, the subgraph of G *induced* by S is defined as $G[S] = (S, E_S)$, where $E_S = \{xy | xy \in E \text{ and } x, y \in S\}$.

A set $D \subseteq V$ is a *dominating set* of $G = (V, E)$ if D dominates V. The concept of domination and its variations are widely studied as can be seen in [15,14]. A dominating set D is called an *outer-connected dominating set* if $G[V \setminus D]$ is connected. The concept of outer-connected domination was introduced by Cyman in [6] and further studied in [1,19–21,24]. A set $D \subseteq V$ is a *total dominating set* of $G = (V, E)$ if D totally dominates V. The Minimum Total Domination (MTD) problem is to find a total dominating set of minimum cardinality of the input graph G. Given a positive integer k and a graph $G = (V, E)$, the Total Domination Decision (TDD) problem is to decide whether G has a total dominating set of cardinality at most k. A survey on total domination is the paper by Henning [16]. The detailed literature on the subject can be found in the recent book [17].

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http://dx.doi.org/10.1016/j.dam.2015.05.009
0166-218X/

Please cite this article in press as: B.S. Panda, A. Pandey, Complexity of total outer-connected domination problem in graphs, Discrete Applied Mathematics (2015), http://dx.doi.org/10.1016/j.dam.2015.05.009

A set $D \subseteq V$ of a graph $G = (V, E)$ is called a *total outer-connected dominating set* of G if D is a total dominating set of G and $G[V \setminus D]$ is connected. The *total outer-connected domination number* of a graph G, denoted by $\gamma_{toc}(G)$, is the cardinality of a minimum total outer-connected dominating set of G. The concept of total outer-connected domination was introduced by Cyman in [7] and further studied in [8,10,13,18]. This problem has applications in computer networks. Consider a client–server architecture based network in which any client must be able to communicate to one of the servers and every server must be able to directly communicate to at least one other server. Since overloading of severs is a bottleneck in such a network, every client must also be able to communicate to another client directly (without interrupting any of the server). A smallest group of servers with these properties is a minimum total outer-connected dominating set for the graph representing the computer network.

The Minimum Total Outer-connected Domination (MTOCD) problem is to find a total outer-connected dominating set of minimum cardinality of the input graph G. Given a positive integer k and a graph $G = (V, E)$, the Total Outer-connected Domination Decision (TOCDD) problem is to decide whether G has a total outer-connected dominating set of cardinality at most k. The Total Outer-connected Domination Decision problem is known to be NP-complete for general graphs and even for bipartite graphs [10].

In this paper, we strengthen the NP-completeness result of the TOCDD problem by showing that this problem remains NP-complete for chordal graphs, split graphs, and doubly chordal graphs. On the positive side, we prove that the TOCDD problem is linear time solvable for bounded tree-width graphs. We also propose a linear time algorithm for computing the total outer-connected domination number of a tree. We prove that the MTOCD problem cannot be approximated within a factor of $(1 - \varepsilon) \ln |V|$ for any $\varepsilon > 0$, unless NP $\subseteq$ DTIME($|V|^{O(\log \log |V|)}$). Finally, we show that the MTOCD problem is APX-complete for graphs with bounded degree 4 and for bipartite graphs with bounded degree 5.

2. Preliminaries

In a graph $G = (V, E)$, the *degree* of a vertex $v \in V$ is $|N_G(v)|$ and is denoted as $d_G(v)$. If $d_G(v) = 1$, then v is called a *pendant vertex*. For a connected graph $G = (V, E)$, a vertex $v \in V$ is called a *cut vertex* of G if $G[V \setminus \{v\}]$ is disconnected. If $G[C]$, $C \subseteq V$, is a complete subgraph of G, then C is called a *clique* of G. A set $S \subseteq V$ is an *independent set* if $G[S]$ has no edge. A graph $G = (V, E)$ is said to be *bipartite* if $V(G)$ can be partitioned into two disjoint sets X and Y such that every edge of G joins a vertex in X to another vertex in Y. A partition (X, Y) of V is called a *bipartition*. A bipartite graph with bipartition (X, Y) of V is denoted by $G = (X, Y, E)$. Let n and m denote the number of vertices and number of edges of G, respectively. Without loss of generality, graphs considered in this paper are connected graphs. A graph $G = (V, E)$ is called a *split graph* if V can be partitioned into two sets, say I and S such that I is an independent set and S is a clique.

A graph G is said to be a *chordal graph* if every cycle in G of length at least four has a *chord*, that is, an edge joining two non-consecutive vertices of the cycle. A vertex $v \in V(G)$ is a *simplicial vertex* of G if $N_G[v]$ is a clique of G. An ordering $\alpha = (v_1, v_2, \ldots, v_n)$ is a *perfect elimination ordering* (PEO) of G if v_i is a simplicial vertex of $G_i = G[\{v_i, v_{i+1}, \ldots, v_n\}]$ for all i, $1 \leq i \leq n$. A graph is chordal if and only if it admits a PEO [11].

Let G be a graph, T be a tree and ν be a family of vertex sets $V_t \subseteq V(G)$ indexed by the vertices t of T. The pair (T, ν) is called a *tree-decomposition* of G if it satisfies the following three conditions:

1. $V(G) = \bigcup_{t \in V(T)} V_t$,
2. for every edge $e \in E(G)$ there exists a $t \in V(T)$ such that both ends of e lie in V_t,
3. $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever $t_1, t_2, t_3 \in V(T)$ and t_2 is on the path in T from t_1 to t_3.

The *width* of (T, ν) is $\max\{|V_t| - 1 : t \in T\}$, and the *tree-width* $tw(G)$ of G is the least width of any tree-decomposition of G [9].

We have the following straightforward observations regarding total outer-connected dominating set.

Observation 2.1. *Let P be the set of vertices adjacent to pendant vertices in a graph G. If D_{toc} is a total outer-connected dominating set of G, then $P \subseteq D_{toc}$.*

Observation 2.2. *Let v be a cut vertex of a graph $G = (V, E)$ such that $G[V \setminus \{v\}]$ has k-components, say, $G_1, G_2, \ldots, G_k$, $k \geq 2$. If v belongs to some total outer-connected dominating set D_{toc} of G, then the vertices of all the components other than one, must be contained in D_{toc}.*

Observation 2.3. *Let v be a pendant vertex of a graph $G = (V, E)$. If D_{toc} is a total outer-connected dominating set of G, then either $v \in D_{toc}$ or $D_{toc} = V \setminus \{v\}$.*

3. NP-completeness results

In this section, we show that the TOCDD problem is NP-complete for chordal graphs, split graphs, and doubly chordal graphs by proposing polynomial reductions from a well known NP-complete problem, called Exact Cover by 3-Sets (X3C) [12] which is defined below.

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