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On-line maximum matching in complete multi-partite graphs with an application to optical networks

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ABSTRACT

Finding a maximum matching in a graph is a classical problem. The on-line versions of the problem in which the vertices and/or edges of the graph are given one at a time and an algorithm has to calculate a matching incrementally have been studied for more than two decades. Many variants of the problem are considered in the literature. The pioneering work (Karp, 1990) considers a bipartite graph where the vertices of one part are revealed one at a time together with their incident edges. In this work we consider maximal d-colorable graphs which are exactly the complete d-partite graphs. The vertices arrive one at a time together with their incident edges, or equivalently with their corresponding colors in some given d-coloring of the graph. We present an optimal 2/3-competitive deterministic algorithm for this on-line problem.

This problem is closely related to that of minimizing the cost of line terminals in star topology optical network. We consider lightpaths arriving in an on-line fashion on a given star network. Our result implies a tight 10/9-competitive algorithm for finding a wavelength assignment minimizing the cost of line terminals in such a network.

1. Introduction

1.1. Background

A matching in a graph is a subset M of its edges such that every vertex of the graph is incident to at most one edge of M. A maximum matching is a matching of maximum size, i.e., having the biggest number of edges among all matchings. The problem of finding a maximum matching of a graph is a century old problem [14].

In the on-line versions of the problem, the input graph is presented incrementally, and an algorithm has to maintain at any given time a matching of the already presented part of the graph. Moreover, the algorithm cannot change its prior decisions when a new portion of the graph is revealed. The on-line bipartite maximum matching problem was introduced in [13]. In that work, the vertices of one part of the bipartite graph arrive in some preselected order together with their neighbors. The greedy algorithm that matches a vertex whenever possible, i.e. whenever a vertex with an unmatched neighbor appears, is 1/2-competitive. Indeed a maximal matching is at least 1/2 of a maximum matching. This was shown to be the best possible for deterministic on-line algorithms even when the graph is bipartite. A $(1 - 1/e)$-competitive randomized algorithm was

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proposed, and was proved later [4,11] to be optimal. The problem was used in various theoretical and application scenarios. This was done, for instance, in [1] for packet switching routing, in [2] for load balancing, and in [3] for QoS networks. The problem was also studied for general weighted graphs in [12], where a 1/3-competitive deterministic algorithm is given.

In this work, whose preliminary version is published in [15], we consider complete multipartite graphs. To the best of our knowledge, this is the first study of on-line maximum matching in these graphs. We note that the size of a maximum matching of a given complete multipartite graph can be calculated easily following [19].

1.2. Our contribution

We consider the on-line maximum matching problem in complete multi-partite graphs. We present an optimal 2/3-competitive deterministic on-line algorithm for this problem. Due to the tight connection between this problem and the problem of line terminal minimization in star topology optical networks, the result implies that the competitive ratio of the latter problem is exactly 10/9. The reader is referred to Section 4 for the details.

In Section 2 we present definitions and preliminary results. The lower bound and upper bound for the competitive ratio of the maximum matching problem are presented in Sections 3.1 and 3.2, respectively. In Section 4 we apply the results to the problem of line terminal cost minimization in optical networks. We conclude and discuss further research directions in Section 5.

2. Preliminaries

2.1. Matchings and maximal d-colorable graphs

A *matching* of a graph $H = (V, E)$ is a subset $M \subseteq E$ such that the maximum degree of the subgraph $H_M = (V, M)$ is at most 1. A vertex is said to be matched (resp. unmatched) by M if its degree in H_M is 1 (resp. 0). The size of a matching is the number $|M|$ of its edges. A maximum matching of H is a matching of H with maximum size.

A (vertex) *coloring* of a graph $H = (V, E)$ is a labeling c of its vertex set such that for every edge $\{u, v\} \in E$, its endpoints u and v are labeled with distinct labels (colors) $c(u) \neq c(v)$. In other words, whenever $c(u) = c(v)$, we have $\{u, v\} \notin E$, i.e. a set of vertices colored with the same color λ induces an independent set $V^{(\lambda)} \stackrel{\text{def}}{=} \{v \in V : c(v) = \lambda\}$ that is termed as the *color class* of λ in the coloring.

A graph is d-colorable (or d-*partite*) if it has a coloring using (exactly) d distinct colors. Clearly, every graph on n vertices is n-colorable. The *chromatic number* $\chi(H)$ of H is the smallest number d such that H is d-colorable. Without loss of generality we assume that the color set of a d-coloring is $[d] = \{1, \dots, d\}$. A graph is *maximal* d-colorable if it is d-colorable and any graph obtained from it by adding a single edge is not d-colorable. The following proposition implies that a graph is maximal d-colorable if and only if it is complete d-partite.

Proposition 2.1. *Let H be a maximal d-colorable graph and c a d-coloring of H. Then:*
(i) *$\{u, v\} \in E$ if and only if u, v belong to distinct color classes of c,*
(ii) *$\chi(H) = d$, and*
(iii) *the coloring c is unique up to a permutation of the colors.*

Proof. (i) Let $H = (V, E)$ be a maximal d-colorable graph, and consider a d-coloring of H with color classes $V^{(1)}, \dots, V^{(d)}$. For every pair of vertices u, v in distinct color classes, we have $\{u, v\} \in E$, because otherwise the graph obtained from H by adding $\{u, v\}$ is d-colorable, contradicting the maximality of H. By definition of coloring, for every pair of vertices u, v in the same color class we have $\{u, v\} \notin E$.

(ii) Clearly, $\chi(H) \leq d$. Moreover, by (i) any set of d representatives of the sets $V^{(1)}, \dots, V^{(d)}$ is a clique of size d. Therefore $\chi(H) \geq d$.

(iii) We show that in any d-coloring c', all the vertices of a set $V^{(\lambda)}$ are colored with the same color. Assume, by way of contradiction, that in some d-coloring c' two vertices $u, v \in V^{(j)}$ are colored with two distinct colors λ, λ'. Consider any set of $d - 1$ representatives of sets $V^{(1)}, \dots, V^{(j-1)}, V^{(j+1)}, \dots, V^{(d)}$. They form a clique, none of the vertices of which can be colored with one of λ, λ'. Therefore c' uses at least $d + 1$ colors, a contradiction. □

Based on the above proposition, the color classes $V^{(1)}, \dots, V^{(d)}$ of a maximal d-colorable graph are uniquely determined. We consider the Maximum Matching in Complete Multipartite Graphs (Mmcm) problem defined as follows:

Mmcm(H) **Input:** A complete multi-partite graph H. **Output:** A matching M of H. **Objective:** Maximize $|M|$.

We use the following lemma when developing our results.

Lemma 2.1 ([19])**.** *Let H be a maximal d-colorable graph on n vertices with color classes $V^{(1)}, \dots, V^{(d)}$. H contains a matching of size $\lfloor \frac{n}{2} \rfloor$ if and only if $|V^{(k)}| \leq \lceil \frac{n}{2} \rceil$ for every $k \in [d]$.*

A maximal d-colorable graph having a matching of size $\lfloor \frac{n}{2} \rfloor$ is termed *balanced*.

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