ARTICLE IN PRESS

Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Refining the complexity of the sports elimination problem

Katarína Cechlárová [a], Eva Potpinková [a], Ildikó Schlotter [b,*]

[a] Institute of Mathematics, Faculty of Science, P.J. Šafárik University, Jesenná 5, 040 01 Košice, Slovakia

[b] Budapest University of Technology and Economics, H-1521 Budapest, Hungary

ARTICLE INFO

Article history:
Received 30 January 2014
Received in revised form 8 January 2015
Accepted 13 January 2015
Available online xxxx

Keywords:
Sports elimination problem
Graph labelling
Parameterized complexity
Multivariate complexity analysis

ABSTRACT

The sports elimination problem asks whether a team participating in a competition still has a chance to win, given the current standings and the remaining matches to be played among the teams. This problem can be viewed as a graph labelling problem, where arcs receive labels that contribute to the score of both endpoints of the arc, and the aim is to label the arcs in a way that each vertex obtains a score not exceeding its capacity. We investigate the complexity of this problem in detail, using a multivariate approach to examine how various parameters of the input graph (such as the maximum degree, the feedback vertex/edge number, and different width parameters) influence the computational tractability. We obtain several efficient algorithms, as well as certain hardness results.

1. Introduction

1.1. Motivation

Imagine we are in the middle of an ice-hockey[1] season. Each participating team has currently a certain score and still some matches to play. Can our favorite team t_0 become a winner of the season? More precisely, given the current scores and the set of remaining matches, is it possible that these matches end in such a way that our team will finish with the maximum score among all teams? If the answer is no, our team is said to be *eliminated*. This is a question that occupies not only players, coaches and managers of teams, but also many sports fans. It has also attracted quite a lot of attention from mathematicians and computer scientists. Papers [1,22] use integer linear programming to solve this problem, but we shall concentrate more on combinatorial approaches, see [4,10–12,15,16,20,23,24].

1.2. Formulation of the problem

Let us suppose that the rules of the game define the set of outcomes of a match as

$$S = \{(\alpha_0, \beta_0), (\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)\}.$$

This formalism corresponds to situations where each match has a 'home' team and an 'away' team, and it can end in any of the $k+1$ ways with the home team getting α_i points and the away team β_i points. For example, $S = \{(0, 1), (1, 0)\}$ for baseball, as this game does not allow draws, and a winning team gets 1 point. Basketball, where the winning team gets 2 points, and both teams in a match that ends in a draw are awarded 1 point, has $S = \{(0, 2), (1, 1), (2, 0)\}$. European football differs from

* Corresponding author.
E-mail addresses: katarina.cechlarova@upjs.sk (K. Cechlárová), eva.potpinkova@student.upjs.sk (E. Potpinková), ildi@cs.bme.hu (I. Schlotter).

1 The reader may substitute any game he or she likes.

http://dx.doi.org/10.1016/j.dam.2015.01.021

basketball in that the winner gets 3 points, so $S = \{(0, 3), (1, 1), (3, 0)\}$ for European football. Examples of other games are given by Kern and Paulusma [16].

A polynomial-time reduction[2] provided also by the same authors [16] showed that we can restrict ourselves to the case where

$$\alpha_0 = 0, \qquad \alpha_1 = 1 < \alpha_2 < \cdots < \alpha_k \quad \text{and} \quad \beta_0 > \beta_1 > \cdots > \beta_{k-1} \geq 1, \qquad \beta_k = 0. \tag{1}$$

The set of outcomes fulfilling (1) is called *normalized*. Throughout the paper we will assume S to be normalized.

An instance of the Generalized Sports Elimination problem with the set S of outcomes (GSE(S) for short) can be described by a triple $(\mathcal{T}, w, \mathcal{M})$. We let $\mathcal{T} = \{t_0, t_1, \ldots, t_n\}$ represent the set of teams participating in the competition. The function $w : \mathcal{T} \to \mathbb{R}$ defines current scores and $\mathcal{M} : \mathcal{T} \times \mathcal{T} \to \mathbb{N}$ the number of remaining matches between teams of $\mathcal{T}$.

By a (t, t')-match for some $t, t' \in \mathcal{T}$ we mean a match played between t and t' such that t is the home team and t' is the away team. The question is whether it is possible that all the remaining matches end in such a way that team t_0 will have the maximum score among all teams. More precisely, given the set of outcomes S, the problem GSE (S) is defined as follows.

Generalized Sports Elimination for S:
Instance: A triple $(\mathcal{T}, w, \mathcal{M})$ as described above.
Question: Can a final score vector $s : \mathcal{T} \to \mathbb{R}$ be reached such that $s(t_0) \geq s(t_i)$ for each $t_i \in \mathcal{T}$?

If the answer is yes, we say that team t_0 is *not eliminated*, otherwise t_0 is *eliminated*. Observe that we can suppose that our team t_0 has already played all its matches, and in each one it obtained the maximum possible score, so its final standing is $w(t_0)$ points. (If in reality this is not the case, we can modify the values of w accordingly).

An instance $(\mathcal{T}, w, \mathcal{M})$ of GSE(S) can quite naturally be represented by a directed multigraph $G = (V, A)$ with vertex capacities $c : V \to \mathbb{R}$. The vertex set $V = \{v_1, \ldots, v_n\}$ of G corresponds to the teams $\mathcal{T} \setminus \{t_0\}$, and arcs $(v_i, v_j) \in A$ correspond to the remaining matches between teams t_i and t_j. More precisely, the multiplicity of an arc (v_i, v_j) equals the number of remaining (t_i, t_j)-matches. The capacity of a vertex $v_i \in V$ is equal to $c(v_i) = w(t_0) - w(t_i)$, and it represents the number of points that team t_i can still win so as not to overcome team t_0. It is easy to see that GSE(S) is equivalent to the following problem that we call Arc Labelling with Capacities for S, or ALC(S) for short.

Arc Labelling with Capacities for $S = \{(\alpha_0, \beta_0), \ldots, (\alpha_k, \beta_k)\}$:
Instance: A pair (G, c) where $G = (V, A)$ is a directed multigraph and $c : V \to \mathbb{R}$ is a vertex capacity function.
Question: Does there exist an assignment $p : A \to \{0, \ldots, k\}$ such that

$$\mathrm{scr}_p(v) := \sum_{a=(v,u)\in A} \alpha_{p(a)} + \sum_{a=(u,v)\in A} \beta_{p(a)} \leq c(v) \tag{2}$$

holds for each vertex $v \in V$?

We say that $p : A \to \{0, \ldots, k\}$ is a *score assignment* for G. If $p(a) = q$ for some arc $a = (u, v) \in A$, then we also say that p assigns the outcome (α_q, β_q) to the arc a, and that u and v *gain* α_q and β_q (resulting) from the arc a, respectively. To keep the notation simple, instead of $p((u, v))$ we shall simply write $p(uv)$. The *score* of some vertex v in p, denoted by $\mathrm{scr}_p(v)$, is defined by the left-hand side of Inequality (2); clearly, $\mathrm{scr}_p(v)$ equals the total points that v gains when all remaining matches yield the outcome as determined by p. We say that a score assignment p for G is *valid* with respect to a capacity function c, if $\mathrm{scr}_p(v) \leq c(v)$ for each vertex $v \in V$. Thus, the task in the ALC(S) problem is to decide whether a valid score assignment exists.

Problem ALC(S) restricted to instances with graphs G having maximum degree at most Δ will be denoted by Δ-ALC(S).

As the reader can see from the definitions of the problems GSE(S) and ALC(S), we take the view that the game (in fact, the set of outcomes S) is fixed, and a different S defines another variant of the elimination problem or of the corresponding graph labelling problem. As a consequence of this assumption, the size of the set S is a constant. However, to guarantee a greater insight into the complexity of the algorithms proposed, we sometimes state running times with their dependence on k made explicit; in all cases where the dependence on k is not explicit, we assume k to be a fixed constant.

1.3. Previous work

If the rules of the game are such that the winner of a match gets 1 point, the loser gets 0 points and there are no draws (like in baseball), that is, $S = \{(0, 1), (1, 0)\}$, then the elimination problem can easily be solved by employing network flow theory. Schwartz [23] was the first one to propose such a method; his network has $O(n^2)$ vertices, where n is the number of teams. Another construction with a network containing only $O(n)$ vertices was proposed by Gusfield and Martel [10].

However, it soon turned out that some score allocation rules make the elimination problem intractable. Bernholt et al. [4] proved that GSE (S) is NP-complete for the European football system where $S = \{(0, 3), (1, 1), (3, 0)\}$. They also mentioned that this result could be generalized to the rules that award α points to the winner and β points to both teams of a match ending in a draw, if $\alpha > 2\beta$. Kern and Paulusma [15,16] extended this result by classifying all score allocation rules S into

[2] The reduction does not change the directed graph underlying the instance (formally defined later on), except for possibly reversing all its arcs.

Please cite this article in press as: K. Cechlárová, et al., Refining the complexity of the sports elimination problem, Discrete Applied Mathematics (2015), http://dx.doi.org/10.1016/j.dam.2015.01.021