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Extending finite-memory determinacy to multi-player games ☆

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ABSTRACT

We show that under some general conditions the finite memory determinacy of a class of two-player win/lose games played on finite graphs implies the existence of a Nash equilibrium built from finite memory strategies for the corresponding class of multi-player multi-outcome games. This generalizes a previous result by Brihaye, De Pril and Schewe. We provide a number of example that separate the various criteria we explore.
Our proofs are generally constructive, that is, provide upper bounds for the memory required, as well as algorithms to compute the relevant Nash equilibria.

1. Introduction

The usual model employed for synthesis are sequential two-player win/lose games played on finite graphs. The vertices of the graph correspond to states of a system, and the two players jointly generate an infinite path through the graph (the *run*). One player, the protagonist, models the aspects of the system under the control of the designer. In particular, the protagonist will win the game iff the run satisfies the intended specification. The other player is assumed to be fully antagonistic, thus wins iff the protagonist loses. One then would like to find winning strategies of the protagonist, that is, a strategy for her to play the game in such a way that she will win regardless of the antagonist's moves. Particularly desirable winning strategies are those which can be executed by a finite automaton.

Classes of games are distinguished by the way the winning conditions (or more generally, preferences of the players) are specified. Typical examples include:

- Muller conditions, where only the set of vertices visited infinitely many times matters;
- Parity conditions, where each vertex has a priority, and the winner is decided by the parity of the least priority visited infinitely many times;
- Energy conditions, where each vertex has an energy delta (positive or negative), and the protagonist loses if the cumulative energy values ever drop below 0;
- Discounted payoff conditions, where each vertex has a payoff value, and the outcome is determined by the discounted sum of all payoffs visited with some discount factor $0 < \lambda < 1$;

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- Combinations of these, such as energy parity games, where the protagonist has to simultaneously ensure that the least parity visited infinitely many times is odd and that the cumulative energy value is never negative.

Our goal is to dispose of two restrictions of this setting: First, we would like to consider any number of players; and second allow them to have far more complicated preferences than just preferring winning over losing. The former generalization is crucial in a heterogeneous setting (also *e.g.* [1,2]): If different designers control different parts of the system, they may have different specifications they would like to enforce, which may be partially but not entirely overlapping. The latter seems desirable in a broad range of contexts. Indeed, rarely is the intention for the behavior of a system formulated entirely in black and white: We prefer a program just crashing to accidently erasing our hard-drive; we prefer a program to complete its task in 1 minute to it taking 5 minutes, etc. We point to [3] for a recent survey on such notions of quality in synthesis.

Rather than achieving this goal by revisiting each individual type of game and proving the desired results directly (*e.g.* by generalizing the original proofs of the existence of winning strategies), we shall provide two transfer theorems: In both Theorem 8 and Theorem 12, we will show that (under some conditions), if the two-player win/lose version of a game is finite-memory determined, the corresponding multi-player multi-outcome games all have finite-memory Nash equilibria. The difference is that Theorem 8 refers to all games played on finite graphs using certain preferences, whereas Theorem 12 refers to one fixed graph only.

Theorem 12 is more general than a similar one obtained by Brihaye, De Pril and Schewe [1], [4, Theorem 4.4.14]. A particular class of games covered by our result but not the previous one are (a variant of) energy parity games as introduced by Chatterjee and Doyen [5]. The high-level proof idea follows earlier work by the authors on equilibria in infinite sequential games, using Borel determinacy as a blackbox [6][3] – unlike the constructions there (cf. [9]), the present ones however are constructive and thus give rise to algorithms computing the equilibria in the multi-player multi-outcome games given suitable winning strategies in the two-player win/lose versions.

The general theme of transferring determinacy results from antagonistic two-player games to the existence of Nash equilibria in multi-player games is already present in *e.g.* [10] by Thuijsman and Raghavan, as well as [11] by Grädel and Ummels. We can distinguish two fundamental approaches here:

In the first, a single win/lose game is constructed from the multi-player multi-outcome game. Intuitively, the first player proposes a putative Nash equilibrium of the multi-player game, and the second player gets to point out a potential deviation. This line of attack was used *e.g.* by Martin to translate Borel determinacy to the existence of values in antagonistic two player Blackwell games [12], and by Bouyer, Brenguier, Markey and Ummels [13] to a variety of graph games. The transfer results here are typically straightforward to state, but using them requires a proof that the winning condition in the derived game is of an appropriate type, which can be difficult or cumbersome.

The second approach uses a large number of simple win/lose games instead. The present paper follows these lines, and thus our transfer results are harder to prove, but easier to use. This approach is also used by Ummels and Wojtczak to study limit-average games in [14]. As a very tame objective, limit-average games sidestep most complicated parts of our proofs, on the other hand, Ummels and Wojtczak also explore concurrent games, which are outside of the scope of this paper.

Echoing De Pril in [4], we would like to stress that our conditions apply to the preferences of each player individually. For example, some players could pursue energy parity conditions, whereas others have preferences based on Muller conditions: Our results apply just as they would do if all players had preferences of the same type.

This article extends and supersedes the earlier [15] which appeared in the proceedings of Strategic Reasoning 2016.

Structure of the paper: After introducing notation and the basic concepts in Section 2, we state our two main theorems in Section 3. The proofs of our main theorems are given in the form of several lemmata in Section 4. The lemmata prove slightly more than required for the theorems, and might be of independent interest for some readers. In Section 5 we discuss how our results improve upon prior work, and explore several notions prominent in our main theorems in some more detail. Finally, in Section 6 we consider as applications two classes of games covered by our main theorems but not by previous work.

2. Background

Win/lose two-player games: A win/lose two-player game played on a finite graph is specified by a directed graph (V, E) where every vertex has an outgoing edge, a starting vertex $v_0 \in V$, two sets $V_1 \subseteq V$ and $V_2 := V \setminus V_1$, a function $\Gamma : V \to C$ coloring the vertices, and a *winning condition* $W \subseteq C^\omega$. Starting from v_0, the players move a token along the graph, ω times, with player $a \in \{1, 2\}$ picking and following an outgoing edge whenever the current vertex lies in V_a. Player 1 wins iff the infinite sequence of the colors seen (at the visited vertices) is in W.

Winning strategies: For $a \in \{1, 2\}$ let $\mathcal{H}_a$ be the set of finite paths in (V, E) starting at v_0 and ending in V_a. Let $\mathcal{H} := \mathcal{H}_1 \cup \mathcal{H}_2$ be the possible *finite histories* of the game, and let $[\mathcal{H}]$ be the infinite ones. For clarity we may write $[\mathcal{H}_g]$ instead of

[3] Precursor ideas are also present in [7] and [8] (the specific result in the latter was joint work with Neymann).

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