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Tropical dominating sets in vertex-coloured graphs

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ABSTRACT

Given a vertex-coloured graph, a dominating set is said to be tropical if every colour of the graph appears at least once in the set. Here, we study minimum tropical dominating sets from structural and algorithmic points of view. First, we prove that the tropical dominating set problem is NP-complete even when restricted to a simple path. Then, we establish upper bounds related to various parameters of the graph such as minimum degree and number of edges. We also give an optimal upper bound for random graphs. Last, we give approximability and inapproximability results for general and restricted classes of graphs, and establish a FPT algorithm for interval graphs.

1. Introduction

Vertex-coloured graphs are useful in various situations. For instance, the Web graph may be considered as a vertex-coloured graph where the colour of a vertex represents the content of the corresponding page (red for mathematics, yellow for physics, etc). Given a vertex-coloured graph G^c, a subgraph H^c (not necessarily induced) of G^c is said to be tropical if and only if each colour of G^c appears at least once in H^c. Potentially, any kind of usual structural problems (paths, cycles, independent and dominating sets, vertex covers, connected components, etc.) could be studied in their tropical version. This new tropical concept is close to, but quite different from, the colourful concept used for paths in vertex-coloured graphs [1,24,25]. It is also related to (but again different from) the concept of *colour patterns* used in bio-informatics [17]. Here, we study minimum tropical dominating sets in vertex-coloured graphs. Some work on tropical connected components and tropical homomorphisms can be found in [4,18]. A general overview on the classical dominating set problem can be found in [21].

Throughout the paper let $G = (V, E)$ denote a simple undirected non-coloured graph. Let $n = |V|$ and $m = |E|$. Given a set of colours $\mathcal{C} = \{1, ..., c\}$, $G^c = (V^c, E)$ denotes a vertex-coloured graph where each vertex has precisely one colour from $\mathcal{C}$ and each colour of $\mathcal{C}$ appears on at least one vertex. The colour of a vertex x is denoted by $c(x)$. A subset $S \subseteq V$ is a

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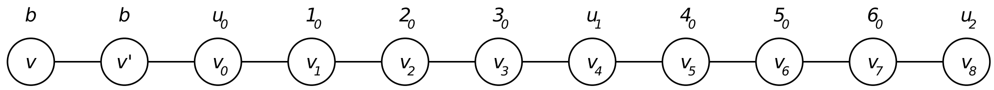

Fig. 1. Example of $\mathcal{P}_0$ when $X = 6$.

dominating set of G^c (or of G), if every vertex either belongs to S or has a neighbour in S. The *domination number* $\gamma(G^c)$ ($\gamma(G)$) is the size of a smallest dominating set of G^c (G). A dominating set S of G^c is said to be *tropical* if each of the c colours appears at least once among the vertices of S. The *tropical domination number* $\gamma^t(G^c)$ is the size of a smallest tropical dominating set of G^c. A *rainbow dominating set* of G^c is a tropical dominating set with exactly c vertices. More generally, a c-element set with precisely one vertex from each colour is said to be a *rainbow set*. We let $\delta(G^c)$ (respectively $\Delta(G^c)$) denote the minimum (maximum) degree of G^c. When no confusion arises, we write γ, γ^t, δ and Δ instead of $\gamma(G)$, $\gamma^t(G^c)$, $\delta(G^c)$ and $\Delta(G^c)$, respectively. We use the standard notation $N(v)$ for the (open) neighbourhood of vertex v, that is the set of vertices adjacent to v, and write $N[v] = N(v) \cup \{v\}$ for its closed neighbourhood. The set and the number of neighbours of v inside a subgraph H is denoted by $N_H(v)$ and by $d_H(v)$, independently of whether v is in H or in $V(G^c) - V(H)$. Although less standard, we shall also write sometimes $v \in G^c$ to abbreviate $v \in V(G^c)$.

Note that tropical domination in a vertex-coloured graph G^c can also be interpreted as "simultaneous domination" in two graphs which have a common vertex set. One of the two graphs is the non-coloured G itself, the other one is the union of c vertex-disjoint cliques each of which corresponds to a colour class in G^c. The notion of simultaneous dominating set[1] was introduced by Sampathkumar [30] and independently by Brigham and Dutton [10]. It was investigated recently by Caro and Henning [11] and also by further authors. Remark that $\delta \geq 1$ is regularly assumed for each factor graph in the results of these papers that is not the case in the present manuscript, as we do not forbid the presence of one-element colour classes.

The Tropical Dominating Set problem (TDS) is defined as follows.

Problem 1. *TDS*
Input: A vertex-coloured graph G^c and an integer $k \geq c$.
Question: Is there a tropical dominating set of size at most k?

The Rainbow Dominating Set problem (RDS) is defined as follows.

Problem 2. *RDS*
Input: A vertex-coloured graph G^c.
Question: Is there a rainbow dominating set?

The paper is organized as follows. In Section 2 we prove that RDS is NP-complete even when graphs are restricted to simple paths. In Section 3 we give upper bounds for γ^t related to the minimum degree and the number of edges. We give upper bounds for random graphs in Section 4. In Section 5 we give approximability and inapproximability results for TDS. We also show that the problem is FPT (fixed-parameter tractable) on interval graphs when parametrized by the number of colours.

2. NP-completeness

In this section we show that the RDS problem is NP-complete. This implies that the TDS problem is NP-complete too.

Theorem 2.1. *The RDS problem is NP-complete, even when the input is restricted to vertex-coloured paths.*

Proof. Clearly the RDS problem is in NP. The reduction is obtained from the 3-SAT problem. Let (I, Y) be an instance of 3-SAT where $I = (l_1 \vee l_2 \vee l_3) \wedge (l_4 \vee l_5 \vee l_6) \wedge \ldots \wedge (l_{X-2} \vee l_{X-1} \vee l_X)$ is a collection of $\tau = X/3$ clauses on a finite set $Y = \{y_1, \ldots, y_m\}$ of boolean variables. From this instance, we will define a vertex-coloured path $\mathcal{P}$ such that $\mathcal{P}$ contains a rainbow dominating set if and only if (I, Y) is satisfiable.

In order to define $\mathcal{P}$, we first construct a segment $\mathcal{P}_0 = vv'v_0v_1 \ldots v_{4\tau}$, and we colour its vertices as follows. Vertices v and v' are coloured black. Vertices $v_0, v_4, v_8, \ldots, v_{4\tau-4}, v_{4\tau}$ are each coloured with a unique colour. The remaining vertices, that will be henceforth called *clausal*, are coloured from v_1 to $v_{4\tau-1}$ with colours $1_0, 2_0, 3_0, \ldots, X_0$. Fig. 1 shows $\mathcal{P}_0$ if $X = 6$.

Next, we define a number of gadgets as follows. If a pair of literals l_i and l_j satisfies that $l_i = \overline{l_j}$, we say that l_j is *antithetic* to l_i. For each literal l_i, $i = 1, \ldots, X$, we consider the list of all literals $l_{i_1}, l_{i_2}, \ldots, l_{i_{k_i}}$ that are antithetic to l_i. Now, to each literal l_{i_f}, $f = 1, \ldots, k_i$, we associate a *constraint gadget* w_{i,i_f} on five vertices defined as follows. Vertex A is an *artificial* one

[1] Also known under the names 'factor dominating set' and 'global dominating set' in the literature.