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Lost in self-stabilization: A local process that aligns connected cells ☆

Damien Regnault [a,*], Éric Rémila [b,*]

[a] IBISC, Univ Evry, Université Paris-Saclay, 91025, Evry, France
[b] Université de Lyon, UJM St-Etienne, GATE Lyon St-Etienne (UMR CNRS 5824), 10 rue tréfilerie, 42023 Saint-Etienne, France

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ABSTRACT

Let t_a and t_b be a pair of relatively prime positive integers. We work on chains of $n(t_a+t_b)$ agents which form together an upper and rightward directed path of the grid $\mathbb{Z}^2$ from $O=(0,0)$ to $M=(nt_a,nt_b)$. We are interested on evolution rules such that, at each time step, an agent is randomly chosen on the chain and is allowed to jump to another site of the grid preserving the connectivity of the chain and the endpoints. The rules must be local, *i.e.*, the decision of jumping or not only depends on the neighborhood of fixed size s of the randomly chosen agent, and not on the parameters t_a, t_b, n. Moreover, the parameter s only depends on t_a+t_b and not on n.

In this paper, we design a rule such that, starting from any chain which does not cross the continuous line segment $[O,M]$, this rule reorganizes this chain into one of the best possible approximations of $[O,M]$. The stabilization is reached after $O((n(t_a+t_b))^4)$ iterations, in average.

The work presented here is at the crossroad of many different domains such as modeling a stabilizing process in crystallography, stochastic cellular automata, organizing a line of robots in distributed algorithms (the robot chain problem) and Christoffel words in language theory.

1. Introduction

1.1. The result

We work on a $2D$ discrete grid of size $A\times B$. We are given a chain of $A+B$ agents, which forms an upper and rightwards directed path between the opposite endpoints of the grid, of coordinates $(0,0)$ and (A,B), such that two successive agents are neighbors. We want to design a rule such that, at each time step, an agent is randomly chosen and is allowed to jump on another site of the grid preserving the connectivity of the chain. The goal of the process is to reorganize the chain in such a way such that it stabilizes into the best possible approximation of the continuous line, of slope $\frac{B}{A}$, passing through the opposite endpoints, *i.e.* the path which is included in a strip of slope $\frac{B}{A}$ with the lowest possible thickness.

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* Corresponding authors.
E-mail addresses: damien.regnault@ibisc.univ-evry.fr (D. Regnault), eric.remila@univ-st-etienne.fr (É. Rémila).

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We present here a distributed algorithm which achieves this goal with very few requirements. All modifications are decided locally by the agents which are memoryless, disoriented, and have a limited range of communication. The decisions only rely on the relative configuration of the closest neighboring agents. The resulting dynamics stabilizes into the desired position in $O((A+B)^4)$ time steps in expectation, up to two conditions.

The first condition concerns the local sight s of each agent (an agent can only observe sites which are at distance at most s from its own position). Let (t_a, t_b) be the pair of relatively prime positive integers such that $\frac{t_b}{t_a} = \frac{B}{A}$. We prove that our process succeeds as soon as $t_a + t_b \leq s$. Informally, this condition says that the algorithm can compute a good alignment vector when its L_1-norm is at most s. We also show that this bound on the sight is almost tight: a sight of at least $t_a + t_b - 1$ is needed to self-stabilize into a discrete line of slope $\frac{B}{A}$.

The second condition is that initially all agents are in the same side of the continuous line linking the two endpoints. When the condition is not satisfied, our algorithm does not completely succeed, because of boundary effects. Nevertheless, we prove that, in this case, the dynamics produces a good approximation of the optimal solution. Moreover, if we avoid boundary effects by identifying endpoints of the chain, (creating a cycle) we also get an optimal approximation of the continuous line.

A simulation of our process is available in the Fig. 14.

1.2. Outline

In Section 2, we define formally our problem. In Section 3, we first present our process and its main properties. The idea of the algorithm is that the selected agent makes a local flip when, locally, this flip does not increase the thickness of the current chain of agents. But finding a local rule which satisfies this condition is not easy. Since the direction of the target line passing by $(0, 0)$ and (A, B) is unknown by the agents, the algorithm has to take into account all the possible directions, and needs to reduce the thickness for each of these directions. This strongly reduces the possibilities of making a flip. The *weak-constraint criterion* is the main tool to define such a process. Moreover, we want the process to have a polynomial average complexity. Thus, we have to forbid the thickness to remain unchanged during a long (exponential) time. So, we forbid some specific flips which may preserve, but not decrease, the thickness. This yields to the *strong-constraint criterion*. This latter constraint allows us to define a process.

Afterward, in Section 4, we introduce an *ad hoc* potential function on chains, inspired by the study of probabilistic cellular automata, and, with this tool, reach our results by a classical use of martingale theory. Finally, in Section 5, we present some questions left open in this paper.

1.3. Motivation

We have previously given a graphical interpretation of our work. Nevertheless, it is at the crossroad of many different domains that we present now.

1.3.1. Crystallography

We were lead to consider this problem when studying a model of stabilizing process of crystals. Crystals are made of several kinds of atoms. At high temperature, the structure of the atoms is chaotic but when the temperature decreases the atoms self-stabilize into a crystal, an ordered structure. Crystals are commonly modeled by tilings [12]. In a set of studies [2,3,9], we developed a model which transforms an unordered tiling into an ordered one to model the cooling of a crystal.

Here, the model represents the cooling of a crystal with two kinds of atoms arranged on a line: one is represented by a horizontal segment and the other by a vertical segment. At high temperature, the segments are unordered and thus the corresponding thread is twisted. In a probabilistic point of view, the distribution of probability on possible configurations is close from the uniform distribution. At low temperature, atoms of the same kind tend to have a regular repartition. At the end of the stabilizing process when the temperature reaches the limit 0, they should be arranged in such a way that their repartition is as much regular as possible. The corresponding configurations in our model is the discretized line linking the two endpoints. We propose here an explanation on how atoms self-stabilize from a chaotic structure to an ordered one by using only local interactions. The fact that the agents of this paper simulate atoms justifies all the previous constraints. All of our precedent studies assume that the number of atoms of each kind remains preserved. In particular, Bodini et al. deal with our problem when the line to approximate has slope 1 (the number of horizontal segments is equal to the number of vertical segments) [3]. The present paper is a generalization of this previous study. This generalization is not direct since, in our model, the target line is unknown. We show that atoms need to consider a wider neighborhood to self-stabilize.

1.3.2. Distributed computing

The robot chain problem [6] in distributed algorithms is really close to our problem except that the topology is continuous. Several solutions were presented to solve this problem but most of them, such as Manhattan-Hopper [13], rely on unavailable information in our setting (robots have names, know global information or can merge together, ...) and thus are not applicable. Nevertheless, one algorithm is interesting for our case: the Go-To-The-Middle algorithm [6]. In this algorithm, when a robot decides to act, it moves to the middle of the line linking its two nearest neighbors. In this paper, the

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