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A new lower bound for the on-line coloring of intervals with bandwidth ☆

Patryk Mikos

Theoretical Computer Science Department, Faculty of Mathematics and Computer Science, Jagiellonian University, Kraków, Poland

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ABSTRACT

The on-line interval coloring and its variants are important combinatorial problems with many applications in network multiplexing, resource allocation and job scheduling. In this paper we present a new lower bound of 4.1626 for the asymptotic competitive ratio for the on-line coloring of intervals with bandwidth which improves the best known lower bound of $\frac{24}{7}$. For the on-line coloring of unit intervals with bandwidth we improve the lower bound of 1.831 to 2.

1. Introduction

An *on-line coloring of intervals with bandwidth* is a two-person game, played in rounds by Presenter and Algorithm. In each round Presenter introduces a new interval on the real line and its bandwidth – a real number from $[0, 1]$. Algorithm assigns a color to the incoming interval in such a way that for each color γ and any point p on the real line, the sum of bandwidths of intervals containing p and colored γ does not exceed 1. The color of the new interval is assigned before Presenter introduces the next interval and the assignment is irrevocable. The goal of Algorithm is to minimize the number of different colors used during the game, while the goal of Presenter is to maximize it.

An *on-line coloring of unit intervals with bandwidth* is a variant of on-line coloring of intervals with bandwidth game in which all introduced intervals are of length exactly 1.

In the context of various on-line coloring games, the measure of quality of a strategy for Algorithm is given by the competitive analysis. A coloring strategy for Algorithm is r-*competitive* if it uses at most $r \cdot c$ colors for any c-colorable set of intervals. The *absolute competitive ratio* for a problem is the infimum of all values r such that there exists an r-competitive strategy for Algorithm for this problem. Let $\chi_A(\mathcal{I})$ be the number of colors used by Algorithm A on the set $\mathcal{I}$ of intervals with bandwidth, and $OPT(\mathcal{I})$ be the minimum number of colors required to color intervals in the set $\mathcal{I}$.

The *asymptotic competitive ratio* for Algorithm A, denoted by $\mathcal{R}_A^\infty$, is defined as follows:

$$\mathcal{R}_A^\infty = \liminf_{k\to\infty}\{\tfrac{\chi_A(\mathcal{I})}{k} : OPT(\mathcal{I}) = k\}$$

The *asymptotic competitive ratio* for a problem is the infimum of all values $\mathcal{R}_A^\infty$ such that A is an Algorithm for this problem.

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E-mail address: mikos@tcs.uj.edu.pl.

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In this paper we give lower bounds on competitive ratios for on-line coloring of intervals with bandwidth and for unit version of this problem. We obtain these results by presenting explicit strategies for Presenter that force Algorithm to use many colors while the presented set of intervals is colorable with a smaller number of colors.

1.1. Previous work

A variant of on-line coloring of intervals with bandwidth in which all intervals introduced by Presenter have bandwidth 1 is known as an on-line interval coloring. The competitive ratio for this problem was established by Kierstead and Trotter [6]. They constructed a strategy for Algorithm that uses at most $3\omega - 2$ colors on ω-colorable set of intervals. They also presented a matching lower bound – a strategy for Presenter that forces Algorithm to use at least $3\omega - 2$ colors. The unit variant of the on-line interval coloring was studied by Epstein and Levy [4]. They presented a strategy for Presenter that forces Algorithm to use at least $\left\lfloor \frac{3\omega}{2} \right\rfloor$ colors. Moreover, they showed that a natural greedy algorithm uses at most $2\omega - 1$ colors.

A variant of the on-line coloring of intervals with bandwidth in which all intervals have the same endpoints is known as the on-line bin packing, see [3] for a survey.

On-line coloring of intervals with bandwidth was first posed in 2004. Adamy and Erlebach [1] showed a 195-competitive algorithm for this problem. An improved analysis by Pemmaraju et al. [8] showed that Adamy–Erlebach algorithm has competitive ratio 35. Narayanaswamy [7] and Azar et al. [2] presented a 10-competitive algorithm, while Epstein and Levy [5] showed a lower bound of $\frac{24}{7}$ for the asymptotic competitive ratio in this problem. On-line coloring of unit intervals with bandwidth was studied by Epstein and Levy [4]. They presented a lower bound of 2 and upper bound of $\frac{7}{2}$ for the absolute competitive ratio in this problem. For the asymptotic competitive ratio, they showed a 3.178-competitive algorithm and a lower bound of 1.831.

1.2. Our result

For the on-line coloring of intervals with bandwidth, we prove that the asymptotic competitive ratio is at least 4.1626. For the on-line coloring of unit intervals with bandwidth, we present an explicit strategy for Presenter that forces Algorithm to use at least $2k - 1$ different colors while the presented set of intervals is k-colorable.

2. Interval coloring

At first we recall a strategy proposed by Kierstead and Trotter for Presenter in the on-line interval coloring game. We use this strategy as a substrategy in our main result.

Theorem 1 *(Kierstead, Trotter [6]). For every $\omega \in \mathbb{N}_+$, there is a strategy for Presenter that forces Algorithm to use at least $3\omega - 2$ different colors in the on-line interval coloring game played on a ω-colorable set of intervals. Moreover, Presenter can play in such a way that every introduced interval is contained in a fixed real interval $[L, R]$.*

Below we present a strategy for Presenter in the on-line coloring of intervals with bandwidth. For a fixed $k \in \mathbb{N}_+$, we ensure that at any point of the game, the set of intervals introduced by Presenter is k-colorable.

Definition 2. A pair of sequences $([j_1, \ldots, j_n], [x_1, \ldots, x_n])$ such that $x_i \in \mathbb{N}_+$, $j_i | k$ and $\forall_{q<i} : j_q < j_i \leqslant \frac{1}{3}k$ is called a *k-schema.*

Not every k-schema describes a strategy for Presenter. Later we give additional conditions that a given k-schema has to satisfy to describe a valid strategy.

The strategy for Presenter based on a k-schema $([j_1, \ldots, j_n], [x_1, \ldots, x_n])$ consists of 2 phases. The first phase, called *separation phase*, consists of n subphases indexed $1, \ldots, n$. Let $\mathcal{M}$ be the set of *marked intervals*, which initially is empty. For each color c used by Algorithm in the separation phase, the set $\mathcal{M}$ contains the first interval colored by Algorithm with c.

All intervals introduced by Presenter in the i-th subphase are contained in the same region $[L_i, R_i]$, have length $s_i = \frac{1}{2}(R_i - L_i)$, and bandwidth $\frac{j_i}{k}$, see Fig. 1. Let l_i be the rightmost right endpoint of a non-marked interval introduced in the i-th subphase, or $l_i = L_i + s_i$ if such an interval does not exist. Let r_i be the leftmost right endpoint of a marked interval introduced in the i-th subphase, or $r_i = R_i$ if such an interval does not exist. For the first subphase set $L_1 = 0$, $R_1 = 2$ and for the i-th subphase $L_i = l_{i-1}$ and $R_i = r_{i-1}$, see Fig. 1, where l_{i-1} and r_{i-1} are values of those variables after the end of the $(i-1)$-subphase.

In the i-th subphase, Presenter introduces new intervals until it gets exactly x_i new colors. Let $p_i = \frac{1}{2}(l_i + r_i)$. A new interval introduced by Presenter has endpoints $I = [p_i - s_i, p_i]$. If Algorithm colors I with one of the already used colors, then l_i changes to p_i. Otherwise, r_i changes to p_i and interval I is marked.

Assume that Presenter constructs some coloring after each subphase. Let Γ_i be the number of colors used by Presenter in that coloring on intervals in the set $\mathcal{M}$ after the i-th subphase, see Fig. 2. In the second phase, called the *final phase*,