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Singularities in structured meshes and cross-fields

Harold J. Fogg [a], Liang Sun [b,*], Jonathan E. Makem [a], Cecil G. Armstrong [b], Trevor T. Robinson [b]

[a] Meshing & Abstraction, Digital Factory, Simulation and Test Solutions Siemens PLM Software, SIEMENS, Francis House, 112 Hills Road, Cambridge, CB2 1PH, UK
[b] School of Mechanical and Aerospace Engineering, Queen's University Belfast, BT9 5AH, UK

A R T I C L E   I N F O

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A B S T R A C T

Singularities in structured meshes are vertices that have an irregular valency. The integer irregularity in valency is called the singularity index of the vertex of the mesh. Singularities in cross-fields are closely related which are isolated points where the cross-field vectors are defined in its limit neighbourhood but not at the point itself. For a closed surface the genus determines the minimum number of singularities that are required in a structured mesh or in a cross-field on the surface. Adding boundaries and forcing conformity of the mesh or alignment of the cross-field to them also affects the minimum number of singularities required. In this paper a simple formula is derived from Bunin's *Continuum Theory for Unstructured Mesh Generation* (Bunin, 2008) that specifies the net sum of singularity indices that must occur in a cross-field with even numbers of vectors on a face or surface region with alignment conditions. The formula also applies to mesh singularities in quadrilateral and triangle meshes and the correspondence to 3-D hexahedral meshes is related. Some potential applications are discussed.

1. Introduction

Structured grids or quadrilateral (quad) meshes with tensor product structure offer numerical advantages over unstructured meshes. However, when it comes to closed surfaces some disruption of the regular grid structure can be necessary. For instance, in a map projection of a globe, singularities in the longitude–latitude grid occur at the poles. When a surface has boundaries forcing the grid to conform to them can also necessitate the introduction of singularities, not just to reduce the distortion of the grid, but there will be a certain minimum number that are essential for facilitating a quad mesh. Although triangle (tri) meshes have a more flexible connectivity than quad meshes, they also have a natural topological structure which must be disrupted for the same reasons.

Cross-fields or N-way rotational symmetry (N-RoSy) vector fields are like vector fields but instead of a single vector being defined at a point they have a set of N equally spread vectors defined. They can be used to describe the 'directionality' of a mesh and their possible singularity combinations are subject to the same conditions as for meshes.

How to determine the requisite cross-field and mesh singularities based on the topological shape of the surface region and the geometric shapes of its boundaries will be covered in this paper.

1.1. Related work

A formula was proposed by White et al. [1] for determining if a face is submappable, which means that a *logical* representation can be found where all edges of the face are horizontal or vertical. First vertices are assigned types, which essentially decides the number of $\pi/2$ turns (signed with respect to an anticlockwise traversal) between the adjacent edges in a local logical representation. Values of 1, 0, -1 and -2 are assigned to *end*, *side*, *corner* and *reversal* vertices (*cf.* Fig. 3). If the sum of the vertex classification values is four it means that the local logical representations of the vertices are consistent with a global logical representation for the face and thus the face is submappable. The check works on the principle that the sum of exterior angles at vertices of a planar polygon must equal 2π.

The formula was reformulated and generalised to multiply connected faces by Ruiz Girones et al. [2]. For the face to be submappable it must have assigned vertex types such that

$$\#End - \#Corner - 2\#Reversal = 4(1 - \#Holes) \tag{1}$$

where *#End*, *#Corner* and *#Reversal* are the number of *end*, *corner* and *reversal* vertex types and *#Holes* is the number of inner boundary loops.

Beaufort et al. [3] have recently presented a general formula for determining the numbers of indices of singularities in tri and quad meshes in consideration of the face's topological shape and

* Corresponding author.
E-mail address: lsun02@qub.ac.uk (L. Sun).

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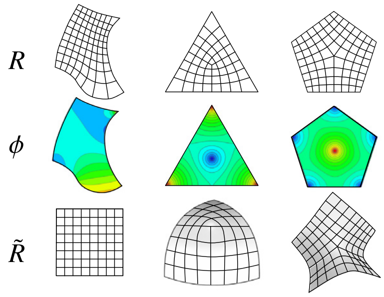

Fig. 1. Illustrations of Bunin's continuum theory for quad meshes on three simple faces.

boundary alignment constraints. Although it is almost equivalent to that shown here, it is derived by a different method of analysing the Euler Formula and topological identities of the meshes and it gives a different perspective of the problem.

Parametrisation-based mesh generation techniques have been intensely developed recently [4,5]. Their approach is to initialise and optimise a cross-field to improve its smoothness and in the process establish the number, indices and placements of mesh singularities. A secondary optimisation method solves for a parametrisation and hence mesh that fits with the cross-field. Palacios et al. [6], Ray et al. [7] and Knöppel et al. [8] prove the Poincaré – Hopf Theorem for cross-fields on smooth and discrete closed oriented surfaces with empty boundaries which relates the net sum of singularity indices (defined slightly differently to here) to its Euler Characteristic.

1.2. Contributions

A simple formula is presented which gives the necessary sum of singularity indices in a cross-field with an even number of vectors. It also applies directly to singularities in quad and tri meshes. The formula is equally simple but is more descriptive to that of Ruiz Girones et al. as it determines the net sum of mesh singularity indices that are required for a face, not just whether the face is submappable, and it accounts for more general face topologies. The formula extends the Poincaré – Hopf Theorem for cross-fields to faces with non-empty boundaries on which alignment constraints are enforced. The formula is shown to be sufficient for cross-fields and necessary but not sufficient for quad and tri meshes. For 3-D hex meshes a new condition to verify line singularity networks is derived.

2. Predicating theories

2.1. Euler characteristic

If the face R is a regular (*i.e.* compact orientable) region of surface S its Euler Characteristic can be defined by

$$\chi(R) = \#V - \#E + \#F, \tag{2}$$

where $\#V$, $\#E$, $\#F$ are the number of vertices, edges and facets of a triangulation (or a polygon mesh) on the face. The value does not depend on triangulation, hence it *characterises* the face [9, Prop. 1, Chp. 4.5]. The Euler Characteristic of a simple face $R = disc$ can be ascertained by analysing a single triangle:

$$\#V = 3, \#E = 3, \#F = 1 \rightarrow \chi(disc) = 1. \tag{3}$$

The relationship between the genus, $g(S)$, and Euler Characteristic, $\chi(S)$, of a closed orientable surface with empty boundary S is

$$g(S) = \frac{2 - \chi(S)}{2} \tag{4}$$

[9, Prop. 4, Chp. 4.5]. The table below gives the genera and Euler Characteristics of familiar closed surfaces.

S	$g(S)$	$\chi(S)$
Sphere	0	2
Torus	1	0
Double torus	2	−2
⋮	⋮	⋮

2.2. Global Gauss–Bonnet Theorem

The Global Gauss–Bonnet Theorem [9, Chp. 4.5] is

$$\sum_{i=1}^{\#C} \int_{C_i} \kappa_g(S)\, ds + \iint_R K(S)\, dA + \sum_{j=1}^{\#C} \gamma_j = 2\pi\,\chi(R) \tag{5}$$

where $C_1 \ldots C_{\#C}$ are smooth regular curves embedded on the surface S that together form the boundary ∂R of the regular region R. Each C_i is positively oriented, *i.e.* the cross-product of the surface normal with a positive-sense-tangent ($\mathbf{n} \times \mathbf{t}$) points into the region. $\gamma_1 \ldots \gamma_{\#C}$ are the exterior angles at boundary vertices between the pairs of adjacent curves of $C_1 \ldots C_{\#C}$. The internal angles θ_j are related to the exterior angles γ_j by

$$\theta_j = \pi - \gamma_j. \tag{6}$$

κ_g denotes the geodesic curvature of the curve C_i and K denotes the Gaussian curvature of the surface.

One insight to be made from Eq. (5) is that by adding in an extra boundary closed loop made from a small circle whose radius $\rightarrow 0$, the LHS goes down by 2π and therefore the Euler Characteristic on the RHS must drop by 1. So, for a face R of a closed orientable surface S with genus $g(S)$ where the boundaries ∂R compose $\#b$ loops that exclude homeomorphic disc regions, the Euler Characteristic of the face is

$$\chi(R) = \chi(S) - \#b, \tag{7}$$

$$= 2 - 2g(S) - \#b. \tag{8}$$

For example, a cylinder can be considered topologically to be a region of a sphere between two boundary loops, therefore,

$$\chi(cylinder) = 2 - 2 = 0. \tag{9}$$

2.3. A continuum theory for unstructured mesh generation in two dimensions

Bunin [10] explains a precise 'continuum theory' of cross-fields and infinitesimal isotropic quad and tri meshes. Amongst other things, it explains the dependency of the number and indices of singularities that must occur on a face due to its intrinsic shape and boundary alignment constraints. Illustrations of the concepts are given in Fig. 1 and their definitions are given in the following sections.

2.3.1. Definition

The continuum theory is derived in terms of a conformal mapping of a region (or face), R, of a surface S to a region, $\tilde{R}$, of a surface $\tilde{S}$. The surface $\tilde{S}$ is locally flat everywhere except at some isolated points referred to as cone vertices, $\tilde{P}_{\text{cone}}$. The region has boundaries, ∂R, composed of curves, C_i, that join at vertices in P_{vertex}. The continuities of the entities are:

- S is of class C^∞,
- $\tilde{S} \setminus \tilde{P}_{\text{cone}}$ is of class C^∞,
- C_i are of class $C^l \mid l \geq 2$.