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Rational offsets of regular quadrics revisited

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ABSTRACT

New lower degree rational parametrizations of regular quadric offsets in the 3-dimensional Euclidean space are explicitly derived. Offsets of ellipsoids, elliptic paraboloids, and doubly ruled quadrics are parametrized with bidegrees (4, 8), (4, 6), and (4, 4), respectively. These degree bounds are expected to be minimal.

1. Introduction

It is not obvious that offsets of regular quadrics in $\mathbb{R}^3$ admit rational parametrizations. Lü (1996) has been the first to prove that the offsets of all regular quadrics are rational. Then the geometric construction for the actual parametrization of this offset was proposed by Peternell (1997) (also shortly described in Krasauskas and Peternell, 2010), which appeared quite involved in the case of ellipsoids.

Recently studying low degree rational patches on isotropic cyclides in Krasauskas et al. (2014) we noticed relations with offsets of quadrics. Oriented tangent planes of a given quadric in $\mathbb{R}^3$ define a surface in the Blaschke cylinder, which is actually an isotropic cyclide. The most complicated cases of such cyclides have two real components. Therefore, $\mathbb{R}$-birational parametrization is not possible, and one can only hope to parametrize both components separately. Finally the required parametrizations were described in Clifford–Bézier formulas in the isotropic space, where control points and weights are elements in particular Clifford algebras.

In this paper our approach is more elementary and geometric. We avoid Clifford algebras here, but use Minkowski space if necessary and quaternions if they allow to simplify formulas. Our focus is on the explicit derivations of parametrization examples rather than on their particular properties or uniqueness issues. The proposed constructions improve the quality and degree bounds of rational offset parametrizations of regular quadrics.

Section 2 is devoted to some notations and necessary details about quadrics, PN-surfaces and Minkowski space. In Section 3 the following offset bidegrees are obtained (quadrics of revolution are excluded): (4, 4) for one-sheeted hyperboloids and hyperbolic paraboloids, (4, 6) for elliptic paraboloids, and (4, 8) for ellipsoids. It seems the obtained degrees are minimal. We formulate this as Conjecture 4.1 in the final section.

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2. Quadrics, PN-surfaces and Minkowski space

In the real projective space $\mathbb{R}P^n$ quadrics are defined by equations $\mathbf{x}Q\mathbf{x}^T = 0$ in matrix form, where $\mathbf{x} = [x_0, \dots, x_n]$ is a row of homogeneous coordinates and Q is a symmetric matrix. Any quadric Q defines the polar correspondence between points in $\mathbb{R}P^n$ and hyperplanes

$$\mathbf{p} \mapsto \hat{\mathbf{p}} : \mathbf{p}Q\mathbf{x}^T = 0, \tag{1}$$

where $\hat{\mathbf{p}}$ can be treated as points $\mathbf{p}Q$ in the dual space $(\mathbb{R}P^n)^*$. The point $\mathbf{p}$ is in the quadric Q if and only if $\hat{\mathbf{p}}$ is a tangent hyperplane to Q at this point. The dual quadric Q^* in $(\mathbb{R}P^n)^*$ is defined as the collection of all such tangent hyperplanes. In general the dimensions of Q and Q^* can be different, as we will see below.

$\mathbb{R}^n$ will be identified with the affine part $x_0 \neq 0$ of $\mathbb{R}P^n$:

$$(x_1, \dots, x_n) \mapsto [1, x_1, \dots, x_n], \quad [x_0, x_1, \dots, x_n] \mapsto (x_1/x_0, \dots, x_n/x_0).$$

For the dual space $(\mathbb{R}P^n)^*$ we take a different choice $y_n \neq 0$ of the affine part: $[y_0, \dots, y_n] \mapsto (y_0/y_n, \dots, y_{n-1}/y_n)$.

We will need some elements of Laguerre geometry. The detailed exposition can be found in Peternell and Pottmann (1998) and Krasauskas and Peternell (2010). Let us consider Euclidean space $\mathbb{R}^3$ embedded as the hyperplane $x_4 = 0$ into $\mathbb{R}^4$ with the Minkowski metric, which is defined by the *absolute quadric* Ω: $x_1^2 + x_2^2 + x_3^2 - x_4^2$ at infinity $x_0 = 0$ in the projective extension $\mathbb{R}P^4$. Points in $\mathbb{R}^4$ will represent oriented spheres. The dual of a nonsingular quadric Q in $\mathbb{R}P^3$ will be $Q^* = Q^{-1}$. However, if we locate Q in the hyperplane $x_4 = 0$ of the ambient space $\mathbb{R}P^4$ then the dual of Q will be 3-dimensional singular quadric in $(\mathbb{R}P^4)^*$ defined by the block diagonal matrix $Q^* = (Q^{-1}, 0)$. It is clear that applying duality once more one gets the initial quadric $(Q^{-1}, 0)^* = Q$. For example, since Ω is in the hyperplane $x_0 = 0$, $\Omega^* = \mathrm{diag}(0, 1, 1, 1, -1)$. It is called the *Blaschke cylinder*. The hyperplanes in $\mathbb{R}^4$ represented by points in Ω^* are called *isotropic hyperplanes*.

Main objects of Laguerre geometry are oriented spheres and oriented planes in $\mathbb{R}^3$. They are represented by points and isotropic hyperplanes in Minkowski space $\mathbb{R}^4$, respectively. Points $\mathbf{x} \in \mathbb{R}^4$ represent spheres with center (x_1, x_2, x_3) and radius x_4, and isotropic hyperplanes $\mathbf{h}$ in $\mathbb{R}^4$ represent planes $\mathbf{h} \cap \mathbb{R}^3$ with unique normals $\mathbf{n}$ defined by their touching points to Ω. Minkowski distance between points coincides with tangential distance between oriented spheres.

A surface in Euclidean space $\mathbb{R}^3$ is a *Pythagorean normal* (PN) surface if it has a rational parametrization $\mathbf{p}(s,t)$ and a rational unit normal $\mathbf{n}(s,t)$ (also called the *Gauss map*). Then the offset surface at oriented distance d can be rationally parametrized

$$\mathbf{p}_d(s,t) = \mathbf{p}(s,t) + d\mathbf{n}(s,t). \tag{2}$$

It is convenient to use dual approach to PN surfaces. Suppose a nonsingular quadric Q admits PN-parametrization $\mathbf{p}(s,t)$ and $\mathbf{n}(s,t)$. Then define a *support function* $h(s,t)$ and oriented tangent planes

$$-h + \langle \mathbf{n}, \mathbf{x} \rangle = 0, \quad h = \langle \mathbf{n}, \mathbf{p} \rangle, \quad \mathbf{x} \in \mathbb{R}^3. \tag{3}$$

Every such oriented tangent plane corresponds to the unique tangent hyperplane in $\mathbb{R}P^4$

$$\mathbf{e}\mathbf{x}^T = 0, \quad \mathbf{e} = [-h, n_1, n_2, n_3, 1] \in (\mathbb{R}P^4)^*, \quad \mathbf{x} \in \mathbb{R}P^4. \tag{4}$$

Actually, the points $\mathbf{e}(s,t)$ belong to the quadric Ω^*, since $|\mathbf{n}| = 1$. Hence the hyperplanes $\mathbf{e}\mathbf{x}^T = 0$ are tangent to both quadrics Q and Ω. Therefore, they define a rational parametrization of the 2-dimensional intersection $Q^* \cap \Omega^*$.

This construction can be inverted. Starting from any rational parametrization $\mathbf{e}(s,t)$ of $Q^* \cap \Omega^*$ one can find points $\mathbf{q} = \mathbf{e}Q^*$ and $\mathbf{e}\Omega^* = [0, e_1, e_2, e_3, -e_4]$ (see (1)) where the hyperplane $\mathbf{e}\mathbf{x}^T = 0$ touches quadrics Q and Ω, respectively. Then PN-parametrization $\mathbf{p}(s,t)$ and $\mathbf{n}(s,t)$ is reconstructed by going to affine coordinates:

$$\mathbf{p} = (q_1/q_0, q_2/q_0, q_3/q_0), \ \mathbf{q} = \mathbf{e}Q^*, \quad \mathbf{n} = (-e_1/e_4, -e_2/e_4, -e_3/e_4). \tag{5}$$

Remark 2.1. Note that if PN-surface is developable then it is represented by a rational curve in the Blaschke cylinder Ω^*. For example, oriented circular cones correspond to conics on Ω^*. They also can be represented as envelopes of linear families of oriented spheres, i.e. by particular lines in Minkowski space $\mathbb{R}^4$.

3. PN parametrizations of quadrics

Our goal is to find a low degree PN-parametrization of a quadric Q in $\mathbb{R}^3$. It follows from Section 2 that in terms of the dual approach this is a problem of rational parametrization of the intersection of two quadrics Q^* and Ω^* in 4-dimensional space. We are going to use families of conics in $Q^* \cap \Omega^*$. According to Remark 2.1 they represent families of oriented circular cones tangent to Q (see Fig. 1). From Krasauskas and Peternell (2010), Peternell (1997) it is known that these conics can appear only as intersections with 2-planes located on the singular quadrics in the pencil $Q^* - \mu\Omega^*$ with the signature $(+,+,-,-,0)$. Therefore, μ should be a root of the equation $\det(Q^* - \mu\Omega^*) = 0$, and the dual quadric