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Extraction of tori from minimal point sets ☆

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ABSTRACT

A new algebraic method for extracting tori from a minimal point set, made of two oriented points and a simple point, is proposed. We prove a degree bound on the number of such tori; this bound is reached on examples, even when we restrict to smooth tori. Our method is based on pre-computed closed formulae well suited for numerical computations with approximate input data.

1. Introduction

The extraction of geometric primitives from 3D point clouds is an important problem in reverse engineering. These 3D point clouds are typically obtained by means of accurate 3D scanners and there exist several methods for performing the 3D geometric primitives extraction. An important category among them are the statistical and iterative methods based on the RANSAC (RANdom SAmple Consensus) paradigm (Fischler and Bolles, 1981; Schnabel et al., 2007; Toony et al., 2015). Key ingredients in this approach are geometric routines that are capable to produce an instance of a given type of shape from a small number of points. For instance, computing the equation of a plane passing through three given points, or passing through a point-with-normal (a point with a normal vector), are basic routines that are intensively used in RANSAC-based methods. In practice, the most used types of shapes are planes, spheres, cylinders, cones and tori. While devising such routines is relatively straightforward for planes and spheres, the cases of cylinders, cones and tori are much more difficult. In a previous work (Busé et al., 2016) we proposed a detailed analysis and efficient algorithms for cylinders and cones. In this note we treat the case of tori. More precisely, we provide a new method for extracting tori from the smallest possible number of conditions, that is to say from two point-with-normal and a single point, which account for seven parameters, i.e. the same number as the degrees of freedom of the considered interpolation problem. We emphasize that it is very important to compute shapes from the smallest possible number of conditions in order to guarantee efficiency and accuracy in these interpolation processes. Most of the methods that are currently used in practical applications for cylinders, cones and tori are based on the solving of overdetermined linear systems so that the computed shapes are not interpolating the point-with-normal data but are only approximating them.

In the sequel, an *oriented point* is a couple of a point and a nonzero vector. A surface is said to interpolate an oriented point if the point belongs to the surface and its associated vector is colinear to the normal of the surface at this point.

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Notice that we are not assuming that the orientation of the normal of the point is the same as the orientation of the surface. Moreover, it is important to deal with inhomogeneous data, that is to say some points are oriented but not all, in order to take into account the estimated accuracy of oriented point clouds that are obtained by means of normal estimation algorithms. A set of data made of points and oriented points is called a mixed set of points.

Previous methods for the extraction of tori from a small number of points in a RANSAC-like approach have been treated from an overdetermined number of conditions, i.e. mixed set of points that are bigger than necessary, which has the consequence that computed tori are not exactly interpolating the data. For instance, in Lukács et al. (1998) tori are approximated from four oriented points, i.e. twelve conditions, and in Kender and Kjeldsen (1991), Kender and Kjeldsen (1995) tori are approximated from three oriented points, i.e. nine conditions. In this note, we propose a new method that interpolate tori from two oriented points and a single point, i.e. from seven conditions, which is precisely the number of parameters needed to instantiate a torus. At the heart of this contribution is an original and subtle modelisation of this interpolation problem, in opposition to the brute force approach that leads to huge and time consuming treatments of polynomial systems of equations of degree 4 in 7 unknowns. Based on this modelisation, our approach relies on adapted algebraic techniques and allows to develop an efficient interpolation algorithm in the context of numerical computations in double precision with approximate data. As a byproduct, we will also get the following theorem of enumerative geometry which seems to be unknown.

Theorem. *There exist at most eight non-degenerated tori (or an infinity) that interpolate three distinct points, two of them being oriented.*

Our strategy of proof relies on geometric constructions related to 3D interpolation of circles which are described in Section 2. Our new torus interpolation method is developed in Section 3. In Section 4, we will report on our numerical experiments, and also show an example for which this upper bound (eight) is reached, even when we restrict to smooth tori.

2. 3D circles passing through two oriented points

The spine of a torus is a 3D circle. Such a circle depends on six parameters, namely three parameters for the coordinates of its center, two parameters for its supporting plane and one last parameter corresponding to its radius. Thus 3D circles and pairs of oriented points share the same number of degrees of freedom, namely six, and hence one may ask how many circles pass through two oriented points.

We recall that a vector V is said normal to a 3D curve $\mathcal{C}$ at a smooth point M when V is orthogonal to the tangent vector to $\mathcal{C}$ at M.

Proposition 1. *We suppose that two distinct points A_1 and A_2 and two nonzero vectors N_1 and N_2 are given. For $i = 1, 2$, we denote by Ω_i the intersection point, possibly at infinity, between the bisecting plane of the segment $[A_1A_2]$ and the line passing through A_i and parallel to N_i (see Fig. 1). Consider the following fitting problem (P): determine all the 3D circles that interpolate A_1 and A_2 and which are normal to N_1 at A_1 and normal to N_2 at A_2.*

(a) *If $\Omega_1 \neq \Omega_2$ and Ω_1, Ω_2 are not both at infinity, then there is one and only one circle that satisfies (P).*
(b) *If $\Omega_1 \neq \Omega_2$ and Ω_1, Ω_2 are both at infinity, then there is no circle that satisfies (P).*
(c) *If $\Omega_1 = \Omega_2$ then there are infinitely many circles that satisfy (P).*

Proof. First, we observe that a 3D circle $\mathcal{C}$ defines a sheaf of spheres whose centers belong to the line L passing by the center of $\mathcal{C}$ and orthogonal to its supporting plane. Suppose that a point A on $\mathcal{C}$ and a nonzero vector N are given. Denote by D the line through A and parallel to N. Then, N is orthogonal to $\mathcal{C}$ at A if and only if N is orthogonal at A to one and only one of the spheres of the sheaf associated to $\mathcal{C}$, namely the one whose center is the intersection point between the lines D and L. We notice that if D and L are parallel lines, i.e. if N is normal to the supporting plane of $\mathcal{C}$, then the corresponding "limit sphere" of the sheaf of spheres has to be seen as the supporting plane of $\mathcal{C}$, since its center is at infinity and its radius is infinite.

Now, returning to our fitting problem, let $\mathcal{C}$ be a circle that interpolates the two distinct points A_1 and A_2. By the previous observation, we have that the vector N_1 is normal to $\mathcal{C}$ at A_1 if and only if N_1 is normal at A_1 to the sphere S_1 whose center is Ω_1 and that goes through A_1. In other words, the vector N_1 is normal to $\mathcal{C}$ at A_1 if and only if $\mathcal{C}$ is contained in the sphere S_1. Similarly, the vector N_2 is normal to $\mathcal{C}$ at A_2 if and only if $\mathcal{C}$ is contained in the sphere S_2 whose center is Ω_2 and that goes through A_2. We recall that if Ω_1, respectively Ω_2, is at infinity then S_1, respectively S_2, is the normal plane to N_1 through A_1, respectively the normal plane to N_2 through A_2.

To conclude the proof we see that if $\Omega_1 \neq \Omega_2$ and Ω_1, Ω_2 are not both at infinity then the intersection of S_1 and S_2 defines a unique circle because this is the intersection of two spheres, or a sphere and a plane, which contains the two distinct points A_1 and A_2. If $\Omega_1 \neq \Omega_2$ and Ω_1, Ω_2 are both at infinity then S_1 and S_2 are two distinct planes whose intersection is the line through A_1 and A_2, so there is no solution to (P) in this case. Finally, if $\Omega_1 = \Omega_2$ then S_1 and S_2 are