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Edge contraction in persistence-generated discrete Morse vector fields

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ABSTRACT

Recently, discrete Morse vector fields have been shown to be useful in various applications. Analogous to the simplification of large meshes using edge contractions, one may want to simplify the cell complex K on which a discrete Morse vector field $V(K)$ is defined. To this end, we define a gradient aware edge contraction operator for triangulated 2-manifolds with the following guarantee. If $V(K)$ was generated by a specific persistence-based method, then the vector field that results from our contraction operator is exactly the same as the vector field produced by applying the same persistence-based method to the contracted complex. An implication of this result is that local operations on $V(K)$ are sufficient to produce the persistence-based vector field on the contracted complex. Furthermore, our experiments show that the structure of the vector field is largely preserved by our operator. For example, 1-unstable manifolds remain largely unaffected by the contraction. This suggests that for some applications of discrete Morse theory, it is sufficient to use a contracted complex.

1. Introduction

Morse theory has become a subject of interest due to its ability to completely describe the flow of a class of vector fields defined over a manifold [1]. Such a description is useful for scientific purposes, as it improves understanding of the behavior of the underlying *Morse function* - a smooth function without any degenerate critical points. This description is called the *Morse–Smale complex*, and it is uniquely defined for a manifold-Morse function pair. A Morse–Smale complex is a decomposition of the manifold into regions of similar flow, called *cells*. Each point on the given manifold is either critical or lies on an *integral line* between a unique pair of critical points. An integral line is a path between (but not including) a pair of critical points with tangent vectors that agree with the gradient of the Morse function at all points. Hence, a cell in the Morse–Smale decomposition is precisely the set of all points in a manifold which lie on an integral line between two given critical points. We include an example of a Morse–Smale complex in Fig. 1. Such a topological characterization of vector fields is important in areas including fluid dynamics and aerodynamics, when it is necessary to work with continuous functions [2].

However, experimental data is not usually collected in the form of Morse functions. This makes applying smooth techniques problematic. The first attempt at constructing a discrete version of the Morse–Smale complex was done by Edelsbrunner et. al. for piecewise linear 2-manifolds, but their techniques are fairly involved [3]. Forman's discrete Morse theory, the focus of this paper, establishes an analog of the Morse–Smale complex for functions defined over cell complexes [4,5]. For simplicity, this paper only considers simplicial complexes. Key to discrete Morse theory is a class of functions called *discrete Morse functions*. Analogous to the smooth case, these functions induce a gradient vector field on their domain. Forman's theory is strictly combinatorial, and most of its results pertain to manipulating discrete vectors. No derivatives are required. Despite this, discrete Morse theory still contains a number of concepts analogous to structures in classical Morse theory. Among these are notions of *critical simplices* and *gradient paths*, which can be thought of as corresponding to critical points and integral lines, respectively. In the discrete case, the highest dimensional cells may serve as local maxima when they are critical, and critical vertices serve as local minima. When critical, all other simplices are the equivalent of saddles. Hence, gradient paths connect higher dimensional simplices to lower dimensional simplices. These gradient paths allow the computation of a Morse–Smale complex for triangulated manifolds, with structure resembling that of the smooth Morse–Smale complex. Forman's theory has found applications in a variety of areas, including cosmology, terrain analysis, and road network reconstruction [6–10].

Discrete Morse functions are defined over all simplices in a simplicial complex, whereas scientific data can usually be thought of as a height function on a set of points. The canonical example of this is terrain modeling: on some collection of points, one has elevation data, but no information on the behavior of the terrain between the sampled points. A simplicial complex can be gener-

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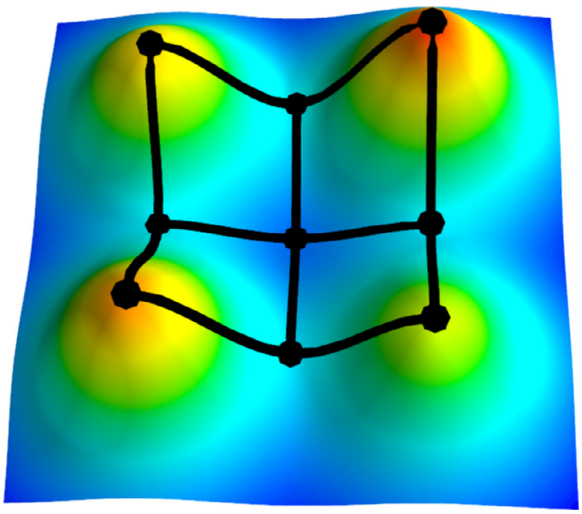

Fig. 1. An example of a smooth Morse–Smale complex with four Morse cells bounded by black curves connecting a maximum (peaks) to two saddles and each saddle to a minimum (pit).

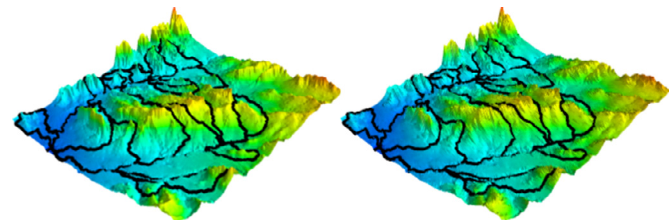

Fig. 2. An example of unstable 1-manifolds on a terrain near Los Alamos, New Mexico before (left) and after (right) 400,000 edge contractions. Note that there is very little difference in the paths of the unstable manifolds.

ated over the terrain by taking some triangulation of the sampled points. These points are the vertices of the simplicial complex, and they are the only simplices in the complex that are associated with a function value. While interpolation methods could be used to assign function values to higher dimensional simplices, this will only be a discrete Morse function on the rarest of occasions. Various techniques have been proposed to compute a discrete Morse vector field from an input height function defined over a complex's vertex set [7,11–14]. We choose to use an algorithm presented by Bauer et. al. which, given any triangulated 2-manifold, outputs a discrete Morse gradient vector field on the triangulation [15]. This algorithm is advantageous in that it is simple (see [10] for further simplification) to implement and very intuitive. It takes as input two parameters: a filtration, or a total ordering on the simplices of K often referred to as $\prec$, and a tuning parameter, δ, which influences the number of remaining critical simplices. When $\delta = \infty$, their algorithm is proven to output a discrete Morse vector field which minimizes the number of critical simplices.

This δ value tunes the output according to the persistence associated with each edge. The idea behind persistence is that if K_0 is chosen to be an empty complex, and K_i is the complex that results after adding the ith simplex under $\prec$ to K_{i-1}, then each added simplex either creates or destroys a homological class. Such a creation or destruction corresponds to a change in the topology of the complex. Persistent homology provides a description of this changing topology by capturing the lifetime – or the "persistence" – of the various classes. For a more thorough treatment of persistent homology, we encourage the reader to consult [16] or [17]. Each edge is then either associated with the persistence of the class which it creates or destroys. The algorithm by Bauer et. al. leaves those edges with persistence $\geq \delta$ as critical.

Triangulated 2-manifolds, together with the discrete Morse gradient vector fields defined on them, can become quite large. An obvious approach to controlling size is to arbitrarily contract edges in the original manifold prior to computing the discrete Morse vector field. Such a procedure is problematic, because a contraction operator that is oblivious to vector field dynamics may lead to a drastically different vector field on the contracted manifold. An alternative is to contract edges in the triangulation and modify the original vector field slightly to fit the new complex. To make contraction as efficient as possible, it is important that the vector field only needs to be modified local to the contraction. Work in this area was initiated by Iuricich and De Floriani, who established such a contraction operator in the context of storing a discrete Morse vector field at several resolutions [18]. However, their criteria for a permissible contraction are fairly strong, in that they disallow circumstances where contraction could be permitted. In particular, they do not permit contractions that destroy critical simplices. This paper establishes a contraction operator which subsumes their criteria and, more importantly, comes with additional mathematical guarantees. If one were to run the algorithm by Bauer et. al. on the contracted manifold with the same δ that generated the original discrete Morse vector field, then the output vector field is guaranteed to be the same as the one which results from the contraction operator. The new operator is established in Section 4, while a formalization of the guarantee is in Section 5.

An *unstable 1-manifold* is the set of all simplices that can be reached by gradient paths originating at a critical edge. An example can be seen in Fig. 2. Various authors have found uses for these manifolds [7–10,13]. These unstable manifolds largely preserve their structure under contraction. Hence, it is often sufficient to use a much coarser complex for the previous applications. In Section 6, we demonstrate this approach on a road network reconstruction application as a proof of concept. In addition, the new contraction operator is more general than the state of the art, and experiments are presented comparing coarsest possible representations. The paper concludes in Section 7 with a discussion on future directions for research.

2. Discrete Morse theory

We now provide background in Forman's discrete Morse theory. For a more thorough treatment, we refer the reader to [4] or [5]. In this section, we define K to be a simplicial complex. Fundamental to discrete Morse theory are *discrete Morse functions*. A function $f: K \to \mathbb{R}$ is a discrete Morse function if it satisfies the following two conditions for all simplices $\sigma \in K$:

$$|\{\tau <_1 \sigma \mid f(\tau) \geq f(\sigma)\}| \leq 1 \quad (1)$$

$$|\{\tau >_1 \sigma \mid f(\tau) \leq f(\sigma)\}| \leq 1 \quad (2)$$

where we write $\tau <_1 \sigma$ or $\sigma >_1 \tau$ if τ is a facet (a face of codimension 1) of σ. Forman proved that both of these quantities cannot be positive for the same simplex.

Lemma 1. *For every simplex* $\sigma \in K$, $|\{\tau <_1 \sigma \mid f(\tau) \geq f(\sigma)\}| = 0$ *or* $|\{\tau >_1 \sigma \mid f(\tau) \leq f(\sigma)\}| = 0$.

These conditions also give a concept of a critical simplex.

Definition 2. A simplex σ is critical if $|\{\tau <_1 \sigma \mid f(\tau) \geq f(\sigma)\}| = 0$ and $|\{\tau >_1 \sigma \mid f(\tau) \leq f(\sigma)\}| = 0$

Critical simplices will play a similar role in computing the Morse–Smale complex as they do in the smooth case.