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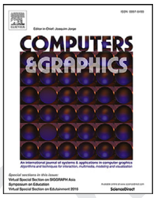

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Hexahedral mesh quality improvement via edge-angle optimization

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We introduce a simple and practical technique to untangle and improve hexahedral (hex-) meshes. We achieve that by enabling the deformation of the boundary surfaces during the untangling process, which provides more space to reach a valid solution. To improve the element quality, an angle optimization strategy is proposed, which has much simpler formulation than the existing method. The deformed volume after optimization is then pulled back to the original one using an inversion-free deformation. In contrast to the current methods, we perform the untangling and quality improvement within a few local regions surrounding elements with undesired quality, which can effectively improve the minimum scaled Jacobian (MSJ) quality of the mesh over the existing method. We demonstrate the effectiveness of our methods by applying it to the hex-meshes generated by a range of methods.

1. Introduction

Hexahedral (or hex-) meshes, are commonly employed by many critical applications that require to solve volumetric partial differential equations. This is mostly due to its naturally embedded tensor product structure, larger tolerance for anisotropy and less numerical stiffness, compared to unstructured meshes (e.g., tetrahedral (or tet-) meshes). These preferred properties enable the convenient imposition of a simulation basis with a higher derivative smoothness between elements of the mesh, and the handling of large deformation during simulations.

However, given any input models, generating hex-meshes with good quality elements while conforming to the surface configuration remains an ongoing challenge. The initially computed hex-meshes, produced by the state-of-the-art methods, such as the polycube mapping or frame-field based methods, often contain inverted elements (i.e., elements with a negative local volume at one or more of its corners), which cannot be directly applied for finite element calculations [1]. Therefore, there is a need for hex-mesh improvement to eliminate the inverted elements and regulate the element shapes [2] while preserving surface features.

A number of techniques have been proposed to untangle and improve hex-meshes with inverted elements without changing their connectivity [2–7]. However, none of them is guaranteed to produce inversion-free hex-meshes. Recently, Livesu et al. [8] introduced an untangling method that optimizes the cone-shapes around the individual edges of the hex-mesh to ensure a positive volume for the tetrahedra around the edges. The formulation of their energy function contains several terms that optimize different geometric characteristics of the mesh. However, the optimization is performed globally with varying weights that prefer elements that already have a good shape. While this strategy helps retain the elements with good quality (i.e. by fixing them), it may prevent the improvement of elements with less optimal quality.

In this work, we propose a local untangling and improvement framework so that the optimization is performed only around inverted elements or elements with quality below a user-specified minimum value (i.e., minimum scaled Jacobian [9], or MSJ). In our local framework, the focus is on improving those elements with undesired quality (i.e., good quality element may become slightly worse), which relieves the stiffness in the global optimization caused by the elements with good quality, allowing the MSJ quality to be further improved. In the meantime, we introduce a new angle-based distortion energy that characterizes different optimization goals (e.g., orthogonality and straightness) via a unified formulation, largely simplifying the setup and solving of the system. Furthermore, to facilitate the search of a valid solution to our optimization, the boundary surface is relaxed if needed. However, relaxing the surface constraint may lead to a large surface distance between the boundary of the output mesh and the original surface. To address that, we perform an inversion-free deformation that gradually pulls the surface back to its original one while still guaranteeing an inversion-free outcome. Note that this inversion-free deformation is only performed after the untangling process. For the improvement of MSJ, this pull-back

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process is not applied, as it may worsen the MSJ – against the goal of MSJ improving. Instead, we directly project the surface back to the original one after improving the MSJ of an inversion-free mesh. After improving the MSJ to a user desired level, we perform a Laplacian-like smoothing to improve the average scaled Jacobian (ASJ) of the mesh. Our framework is simple to implement and can handle more challenging inputs than the existing methods. In average, our method takes 2 minutes for a mesh with 10k-20k elements. We have applied our method to over 80 meshes generated by the polycube-based methods, octree-based method, and frame-field based method , respectively, to demonstrate its effectiveness. All our results have been submitted as the supplemental material, and a reference implementation will be released upon acceptance.

2. Related work

In this section, we review the most relevant literature for the creation and optimization of hex-meshes.

Hex-meshing. Considering its importance to finite element simulation [10], a large amount of effort has been dedicated to the generation of valid all-hex meshes. These methods range from the early sweeping and paving [11,12] grid-based [13–16] and octree-based methods [17–20] to the polycube-based [21–24] and frame-field based approaches [25–28]. A recent survey [29] provides a detailed look at the advances in this direction. Despite these many existing hex-meshing techniques, most initial hex-meshes generated with these approaches need to undergo a quality optimization process to substantially improve their quality for practical use. Our method can be used to optimize the initial meshes produced by a variety of these methods.

Hex-Mesh Optimization. Since it is a necessary step in the meshing pipeline, an equally large amount of work for the improvement of the hex-mesh quality has been proposed. There are two different strategies to improve the mesh quality. The first strategy adopts various smoothing (e.g., the Winslow smoothing [30]) and optimization methods (e.g., via the geometric flow [31]) to optimize the mesh without changing its connectivity, while the second strategy requires the modification of the mesh connectivity to achieve the desired quality improvement, such as the padding process [18,32] typically used in the polycube-based methods. Other methods, like the singularity alignment [33] and polycube domain simplification [34,35] have been proposed to optimize the structure of the hex-meshes. Our method belongs to the first group.

In order to optimize the quality of a hex-mesh, a quality metric has to be identified for the optimizer to improve upon the mesh. According to a Sandia Report by Stimpson et al. [9], there are more than a dozen quality metrics for hex-meshes. Most of these quality metrics measure the difference between a given hexahedron and a canonical cube via either angle distortion, length ratio or tensor distortion. Although there is not a comprehensive study on the effectiveness of these metrics [36], the scaled Jacobian metrics are the most commonly used metrics in the meshing and simulation communities. Intuitively, the Jacobian metric measures the solid angle distortion at the corners of a hexahedron. If the solid angles at the corners are all 90°, the scaled Jacobian achieves the optimal value of 1. It is well-known that a hexahedron can be decomposed into eight overlapping tetrahedra. It may be natural to use various tet-mesh optimization techniques [37,38] to optimize these individual tetrahedra. It is also worth noting that many simplicial and polygonal map optimization techniques [39–41] can also be applied to optimize tet-meshes. However, as already shown in the work by Livesu et al. [8], simply optimizing the tetrahedra associated with the corners of a hexahedron may not improve its quality. Fu et al. [42] introduced an advanced MIPS method for computing locally injective mappings, which can be used to substantially improve the quality of a couple hex-meshes. However, only a few simple hex-meshes with no inverted elements were used in their testing. It is unclear how general this can be when applied to other hex-meshes with a substantially lower quality.

Besides that, many other hex-mesh optimization techniques exist. As reviewed by Wilson [43] and Livesu et al. [8], these techniques generally focus on untangling inverted elements (i.e., with negative scaled Jacobian) and improving the average element quality. Knupp introduced techniques to untangle the inverted elements [2] and improve the overall quality of the hex-mesh [3], which later have been integrated into the famous Mesquite library [4]. Specifically, the Mesquite library attempts to simultaneously untangle and improve the hex-mesh by minimizing an ℓ_1 function. However, since it optimizes one vertex at a time, the performance of Mesquite is slow when applied to hex-meshes with a large number of inverted elements. Later methods resort to local Gauss–Seidel approaches to iteratively untangle and smooth meshes [5–7]. Besides the Gauss–Seidel optimization strategies, non-linear optimization has also been applied to improve the hex-mesh quality [43]. Other optimization techniques for specific types of hex-meshes also exist, such as the quality improvement method for octree-based hex-meshes by Sun et al. [44]. Like many existing approaches, our method can handle hex-meshes generated by different methods (Section 4).

Recently, Livesu et al. [8] introduced the edge cone descriptor that indirectly measures the distortion of the hexahedra via a set of tetrahedra around each mesh edge. Based on this descriptor, a non-linear energy function is defined globally. To solve it, a local-global strategy is applied. As shown by the authors, this approach can untangle meshes that previous methods may fail to untangle. Therefore, we consider this method state-of-the-art and compare our method with it in this paper.

3. Methodology

Similar to many mesh optimizers, given an input mesh with a valid all-hex connectivity, our method first corrects the inverted elements, then improves the overall mesh quality. We also allow the boundary vertices to move out of the original volume if a valid solution cannot be found during untangling. This relaxation alleviates the difficulty of untangling elements at the concave areas of the surface. However, different from most methods, we directly measure the distance of the angles between pairs of connected edges from their respective ideal angles, leading to an intuitive and unified distortion energy formulation. In summary, our method consists of the following key steps (Fig. 1).

Compute target surface. In this step we improve the quality of the surface and associate surface vertices with the features detected from the input mesh (Section 3.1).

Untangling. We detect all inverted elements based on their scaled Jacobians. A local optimizer coupled with a surface relaxation strategy is then used to untangle those inverted elements iteratively until an inversion-free outcome is obtained (Section 3.2).

Inversion-free volume deformation. Due to the relaxation of surface constraint, after the above untangling process, the boundary surface of the output inversion-free hex-mesh may be far away from the original surface. We then perform an inversion-free deformation to pull the current surface back to its original one procedurally (Section 3.3). This step is optional, most models do not need this step.

Improve MSJ. Even though the mesh is currently inversion-free (i.e., all elements have positive scaled Jacobian), its MSJ may still be too low for practical use. To further improve the MSJ, we adopt the above untangling process but with a larger target MSJ (>0) set by the user and perform the same local optimization (Section 3.4).

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