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A polyhedral study of the cardinality constrained multi-cycle and multi-chain problem on directed graphs

Vicky Mak-Hau

School of Information Technology, Deakin University, Waurn Ponds, Geelong, Victoria 3216, Australia

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ABSTRACT

In this paper, we study the Cardinality Constrained Multi-cycle Problem (CCMcP) and the Cardinality Constrained Cycle and Chain Problem (CCCCP). A feasible solution allows one or more cardinality-constrained cycles to exist on the digraph. A vertex can only be involved in at most one cycle, and there may be vertices not involved in any cycles. The CCCCP has an additional set of vertices that can only serve–as the starting vertex of a chain. Apart from cycles, a feasible solution to the CCCCP may also contain multiple cardinality-constrained chains. A vertex can be involved in a chain or a cycle, but not both. Both of the CCMcP and the CCCCP are NP-hard.

This paper focuses on the polyhedral study of the arc-based formulations for both problems. We prove that 3 classes of constraints are facet-defining for the CCMcP polytope, identify 4 new classes of constraints and prove their validity. We then prove that the non-negativity and the degree constraints are facet-defining for the CCCCP polytope. Even though we cannot expect to find a complete polyhedral description (CPD) of the CCMcP or the CCCCP, as both problems are NP-hard, any partial description is always interesting for both theoretical and computational purposes, since the wider the linear description, the less need for branching. A CPD is composed of facet-defining constraints, hence the major contribution of this paper is one step towards finding a CPD for the CCMcP and the CCCCP. We tested the strengths of the facet-defining constraints and new valid constraints on two sets of randomly generated data instances. We reported the numerical results and discussed future research directions.

1. Introduction

We first provide a mathematical description of the two combinatorial optimization problems under study in this paper. Consider a digraph $D=(V,A)$ with V the set of vertices and A the set of arcs. Each arc $a\in A$ is associated with a weight w_a. A feasible solution to the Cardinality Constrained Multi-cycle Problem (CCMcP) is a set of arcs forming several (maybe zero) vertex-disjoint cycles. Note that there may be vertices not involved in any cycle and that the empty solution is also considered feasible. A k-cycle is a single-cycle that involves k arcs (and vertices). The cycles in a CCMcP are constrained in size, with cardinality not exceeding K, for $2\le K\le |V|$. An optimal solution to the CCMcP, however, is one that maximizes the total weight of arcs involved in all cycles. In the case when all arc weights are “1”, the objective function is then equivalent to maximizing the total number of arcs used. In comparison to the well-known Asymmetric Travelling Salesman Problem (ATSP), the two main differences are: (1) in an ATSP, the Assignment Problem (AP) relaxation also allows multiple cycles (subtours), but these cycles are not constrained in size, however “subtours” in a CCMcP are; and (2) all vertices in an ATSP must be visited, but this is not the case for a CCMcP.

Now we describe the Cardinality Constrained Cycle and Chain Problem (CCCCP). Consider again $D=(V,A)$, where the set of vertices V is partitioned into: $V=N\cup P$ with $N\cap P=\emptyset$, and the set of arcs is given by: $A=\{(i,j)\mid i,j\in P, i\neq j\}\cup\{(i,j)\mid i\in N, j\in P\}$. A feasible solution to the CCCCP, like that of a CCMcP, allows one or more cardinality constrained cycles, though a cycle can only involve arcs in the set $\{(i,j)\,|\,i,j\in P, i\neq j\}$. A CCCCP also allows one or more chains, where the first arc in a chain must be from the set $\{(i,j)\,|\,i\in N, j\in P\}$, and subsequent arcs must be from the set $\{(i,j)\,|\,i, j\in P, i\neq j\}$. An ℓ-chain is a chain that contains ℓ vertices. The length of the chains is constrained to be no more than L vertices. A vertex in P cannot be in more than one cycle or chain, and a vertex in N cannot be in more than one chain. An optimal solution to the CCCCP is one that maximizes the total weight of arcs in all chains and cycles.

The CCMcP and the CCCCP have been studied in the context of Kidney Exchange Problems (KEPs) (which is under the general

E-mail address: vicky.mak@deakin.edu.au

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umbrella of barter exchange) both in terms of mathematical modelling and solution methodologies. In the literature of KEPs, $L = K$ is considered in e.g., Manlove and O'Malley (2012); $L > K$ is considered in, e.g., Glorie et al. (2014), Dickerson et al. (2016b) and Plaut et al. (2016); and $L = \infty$ in, e.g., Anderson et al. (2015). A review of KEP solution methods will be provided in Section 1.2. The CCMcP is NP-hard, except for the case when $K = 2$ (see, e.g., Abraham et al., 2007 and Biró et al., 2009). The CCCCP is also NP-hard (see, e.g., Anderson et al., 2015).

The main contribution of this paper is the theoretical analysis of the arc-based formulations of the CCMcP and CCCCP, with the aim of adding to the literature of polyhedral analyses of arc-based formulations of constrained or unconstrained single- or multi-cycle problems defined on directed or undirected graphs. As both of the CCMcP and the CCCCP are NP-hard, we cannot expect to find a complete polyhedral description for either of them. However, any partial description is always interesting for both theoretical and computational purposes (the wider the linear description, the less need for branching). In Section 1.3, we review polyhedral results on a number of closely related combinatorial optimization problems. To the best of our knowledge, there has not been any polyhedral study of the CCMcP or the CCCCP, except for Mak-Hau (2015).

1.1. The kidney exchange problem

The Kidney Exchange-family of Problems (KEPs) have attracted the attention of the combinatorial optimisation community in the last decade or so. The KEP can be represented on a directed graph, with vertices representing the incompatible patient-donor pairs (PDPs)–by an incompatible pair, we mean a patient-donor pair (usually family or friends) such that the patient cannot accept the kidney of the donor due to ABO blood type incompatibility or positive serological cross match. A kidney exchange pool contains a large number of such PDPs. If the kidney of the donor in Pair A is a match for the patient in Pair B, then it can be represented as an arc from vertex A to vertex B on the digraph, and if an arc also exists in the opposite direction, then an exchange of kidneys can be carried out between PDPs A and B. Such an exchange is called a 2-cycle (a cycle involving 2 PDPs, hence 2 vertices on the digraph). An exchange can also involve more than two vertices. A 3-cycle may have the kidney of the donor in Pair A be donated to the patient in Pair B, that of the donor in Pair B donated to the patient in Pair C, and that of the donor in Pair C donated to the patient in Pair A. A k-cycle will be a single cycle that involves k pairs of PDPs, with no sub-cycles involved. As a donor is not legally bound to donate a kidney, in order to avoid donors quitting the program as soon as their partners receive their kidneys, the exchanges involved in a k-cycle are normally performed simultaneously, hence it is impractical for the cardinality limit (K) of the kidney exchange cycles to be too large. In the context of kidney exchange, 2- and 3-cycles are very common, although the largest exchange cycle performed involved nine PDPs (see, e.g., SF Gate, 2015). In a kidney exchange solution, multiple exchange cycles exist in a pool, hence the underlying combinatorial optimisation problem is in fact a CCMcP.

In recent years, some kidney exchange pools have altruistic donors involved. A kidney exchange sequence that begins from an altruistic donor, who donates his/her kidney to a patient in a PDP, with the donor of this PDP in turn donating his/her kidney to the patient in a another PDP and so on, will eventually terminate at a deceased donor waiting list. Such a sequence of kidney exchanges is called a chain. The value L in a CCCCP is called the *cap size* in the context of KEP. On a directed graph, we will use the set N to denote the set of altruistic donors, and P to denote the set of PDPs. A kidney donate chain that begins from an altruistic donor will form a chain on the directed graph. As both chains and cycles are expected to exist as a solution to a kidney exchange optimisation problem, the underlying combinatorial optimisation problem is a CCCCP.

The objective of a kidney exchange optimisation problem is to maximise either the number of kidney exchanges, or a weighted sum of some metrics of the exchanges, and by weight, we mean a "score" for each transplant based on some prioritisation scheme. This means that in the underlying combinatorial optimisation problem, the objective function is to either maximise the total number of arcs used in the cycles (and chains) or to maximise the total weighted arc costs.

1.2. State-of-the-art solution methods

Integer programming models developed for the CCMcP and the CCCCP, (which are mostly developed in the context of KEP), can be classified into three main branches: (a) arc-based models with a small number of variables, but exponentially many constraints (see, e.g., Roth et al., 2007 for CCMcP and Mak-Hau, 2017 for CCCCP); (b) cycle-based models with a small number of constraints, but exponentially many variables (see, e.g., Abraham et al., 2007 and Roth et al., 2007 for CCMcP and Anderson et al., 2015 for CCCCP); and (c) arc-based compact models that create multiple clones of the directed graph, making it possible to have both variables and constraints be polynomial in size (see, e.g., Constantino et al., 2013 and Dickerson et al., 2016b). In terms of solution methodologies, branch-and-price based methods are applied with implementation details presented in, e.g., Abraham et al. (2007), Glorie et al. (2014), Klimentova et al. (2014) and Plaut et al. (2016). A summary review of the performances of Abraham et al. (2007), Manlove and O'Malley (2012), Constantino et al. (2013), Glorie et al. (2014), Klimentova et al. (2014), Anderson et al. (2015), and Mak-Hau (2017) can be found in Mak-Hau (2017), together with details on the sizes of problem tackled, and values K and L tested.

Recently, Dickerson et al. (2016b) presented a new polynomial size formulations for the CCMcP and the CCCCP with bounded L wherein a binary variable is used to determine whether an arc is used in a particular position of a cycle in a particular copy of the digraph–a concept that is an extension to the extended edge formulation proposed by Constantino et al. (2013) and with a stronger LP relaxation (LPR) bound. A polynomial size algorithm for variable elimination in pre-processing was discussed and implemented. The position-indexed edge formulation (PIEF) is developed for the CCMcP. For the CCCCP, a similar idea is used for the chain variables, though for the cycles, a binary variable is used for each cycle. When K is small, these cycles can be completely enumerated, but a pricing algorithm was proposed for a branch-and-price version of the model, called PICEF. Finally, a hybrid version is proposed that uses the position based binary variables for both cycles and chains (HPIEF). The authors extensively tested their methods on a number of large-scale problem instances, with $K = 3$ and L varying from 2 to 12, and sizes as large as, e.g.: $|P| = 700$ with $|N| = 35$, and $|P| = 500$ with $|N| = 125$. From the experiments, it appears that PICEF and HPIEF outperformed all other methods tested. The paper also provided a proof that the LPR of PIEF is as strong as the cycle-formulation (see Abraham et al., 2007 and Roth et al., 2007).

The models/methods listed above are mainly for single-objective implementation. Multi-objective approaches are discussed in Glorie et al. (2014), and implemented in Manlove and O'Malley (2012) and Manlove and O'Malley (2015). The method of Manlove and O'Malley (2012) and Manlove and O'Malley (2015) promote the use of shorter cycles e.g., 2-cycles as well as 3-cycles with a back arc (i.e., the subgraph of the 3-cycle contains at least one 2-cycle, such that if one arc is broken, i.e., one transplant cannot move forward, the remaining two PDPs