European Journal of Operational Research 000 (2018) 1–11

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Estimation and asymptotics for buffered probability of exceedance

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ARTICLE INFO

Article history:
Received 14 December 2016
Accepted 8 January 2018
Available online xxx

Keywords:
Stochastic programming
Buffered probability of exceedance
Sample average approximation
Rare events
Minimum volume ellipsoid

ABSTRACT

This paper studies statistical properties of empirical (sample) estimates of the buffered probability of exceedance (bPOE). The estimation procedure is based on one dimensional minimization representation of the bPOE. Convergence rates and asymptotic properties of the suggested estimation procedures are investigated. Theoretical predictions are validated with numerical experiments, including a special case of exponential distribution, and a study proposing bPOE modification of minimum volume ellipsoid problem.

1. Introduction

For a given probability distribution of losses and a threshold value, *Probability of Exceedance* (POE), is defined as the probability that the random variable of loss exceeds the threshold. This is a natural measure of uncertainty in losses. The POE is very popular in various engineering applications. For instance, nuclear engineering considers probability that radiation release will exceed specified level, while structural reliability analysis considers probability that load exceeds some threshold. Although POE is included in government regulations, it has some major shortcomings. From a conceptual point of view, the threshold in POE provides a low bound on tail outcomes exceeding this threshold. However, POE does not provide information about the magnitude of these outcomes. In other words, POE is capable of registering an exceeding outcome, but incapable of measuring its impact on the system. Also POE has troublesome mathematical properties for discretely distributed random variables, which are typically obtained from sample data. For these variables, POE is discontinuous with respect to the threshold, which prevents using standard sensitivity analysis based on derivatives. In addition, POE is difficult to optimize because optimization problems for POE are usually reduced to large-scale Mixed-Integer programming involving binary variables, a problem that may be hard to solve.

Buffered Probability of Exceedance (bPOE) for a random variable is a counterpart of the POE. The notion of bPOE was introduced and studied in Mafusalov and Uryasev (2014) and in Norton and Uryasev (2014). For a specified threshold, bPOE equals the probability of an upper tail of the distribution, such that the average of this tail coincides with the threshold. There is a similarity between POE and bPOE: the values of bPOE and POE are bounded between zero and one, and, for a given random variable and varying threshold, decrease with the threshold increase. However, bPOE is an upper bound for POE because it includes all outcomes exceeding the threshold, as well as some outcomes below the threshold. The outcomes below the threshold form the so called buffer, therefore, bPOE is a buffered POE. In that sense, the estimate of loss uncertainty given by bPOE is more conservative then the one given by POE.

The tail averages for probability distributions were introduced by Rockafellar and Uryasev (2000) by employing the notion of Conditional Value-at-Risk (CVaR). Specifically, $\mathrm{CVaR}_{1-\alpha}$ defines average in the upper α-tail of a probability distribution. Therefore, bPOE for a random variable X at a threshold $x \in \mathbb{R}$ equals α such that

$$\mathrm{CVaR}_{1-\alpha}(X) = x.$$

In this sense it is said that bPOE is an inverse function of CVaR. Some attractive mathematical properties hold for bPOE. It is continuous in threshold x (maybe except at one point) and quasi-convex in X. Furthermore, it was proved that bPOE is the tightest upper bound for POE among functions consistent with convex

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[1] Research of these authors was partially supported by the USA AFOSR grants: "Design and Redesign of Engineering Systems", FA9550-12-1-0427, and "New Developments in Uncertainty: Linking Risk Management, Reliability, Statistics and Stochastic Optimization", FA9550-11-1-0258.
[2] Research of this author was partly supported by NSF grant 1633196 and DARPA EQUiPS program, grant SNL 014150709.

https://doi.org/10.1016/j.ejor.2018.01.021
0377-2217/

Please cite this article as: A. Mafusalov et al., Estimation and asymptotics for buffered probability of exceedance, European Journal of Operational Research (2018), https://doi.org/10.1016/j.ejor.2018.01.021

stochastic dominance, see Section 3.4 in Mafusalov and Uryasev (2014). One way to interpret this result is that when decision making with POE criterion is preferred, but also the decision maker is risk averse, then bPOE provides the closest suitable criterion. Moreover, a problem of bPOE minimization can be reduced to convex and even linear programming.

Mafusalov and Uryasev (2014) provide a detailed description of mathematical properties of bPOE and various optimization problem statements. When it comes to formulating optimization problems, the connection of bPOE and CVaR provides additional insights. In particular, CVaR and bPOE level constraints are equivalent. Furthermore, using CVaR or bPOE as objective leads to two parametric optimization problem families, and these families, with minor exceptions, share frontiers of optimal solutions. That is, CVaR minimization solution is not found directly from a single bPOE minimization solution, but rather it is found from a collection of solutions, and vice versa.

Given the above connections, it might seem that introduction of bPOE in optimization is redundant. However, this is only true when needs of decision maker require establishing the whole frontier of CVaR/bPOE optimal solutions by solving multiple (and, in general, infinitely many) optimization problems. Not all practical problems possess the luxury of investigating entire solution frontiers prior to making a decision. On the contrary, oftentimes computational resources are sufficient for solving a single optimization problem. Then, a choice between bPOE and CVaR comes from a choice of parameter type, which should be dictated by the nature of the problem or the motivation of decision maker. That is, some applications relate best to a specified fraction of worst cases, then CVaR objective is used; other applications relate best to a specific loss value threshold, then bPOE is used.

The bPOE concept is an extension of the *Buffered Failure Probability* suggested by Rockafellar (2009) and explored by Rockafellar and Royset (2010). The connection is such that buffered failure probability equals bPOE at zero threshold value. Buffered failure probability is built to be aligned with *failure probability*, defined as the probability that system fails, which happens when a corresponding random variable takes a positive value.

This paper studies statistical properties of empirical (sample) estimates of bPOE. The estimators are based on one-dimension minimization representation of bPOE suggested in Mafusalov and Uryasev (2014) and in Norton and Uryasev (2014). In particular, the asymptotic convergence of the suggested estimators is studied.

This paper is organised as follows. Section 2 formally discusses bPOE and some of its properties, introduces necessary notations, and proves results on bias, asymptotical variance, and convergence for a sample estimate of bPOE. Section 3 discusses approaches, including the importance sampling method, for estimating bPOE in case of rare events. The theoretical results of Sections 2, 3 are validated with numerical experiments in Section 4, where a special case of exponential distribution is considered. Convergence properties for optimal solutions and optimal values for bPOE minimization problem are derived in Section 5. A modification of minimum volume ellipsoid (MVE) problem is considered in Section 6. There, instead of minimizing a fraction of non-covered samples under a covering ellipsoid volume constraint, it is proposed to minimize bPOE with the same constraint. The resulting problem is convex, and hence can be efficiently solved. In a case of true sample generating distribution being elliptical, solutions for POE and bPOE minimization coincide. Theoretical results on optimal solution and value convergence are validated for the considered bPOE-modification of the MVE problem. This approach is closely related and is alternative to the conditional MVE by Gotoh and Takeda (2006, 2008).

2. Statistical properties of buffered probability estimates

For $\alpha \in [0,\ 1)$ Conditional Value-at-Risk (also called Average Value-at-Risk, Expected Shortfall and Expected Tail Loss) of a random variable X is defined as[3]

$$\mathrm{CVaR}_\alpha(X) := \inf_{t\in\mathbb{R}} \left\{ t + (1-\alpha)^{-1}\mathbb{E}[X-t]_+ \right\}. \tag{2.1}$$

Here and further, we assume that $\mathbb{E}|X| < \infty$, and hence the expectation in (2.1) is well defined and finite valued. For $\alpha = 0$, $\mathrm{CVaR}_0(X) = \mathbb{E}[X]$ and $\mathrm{CVaR}_\alpha(X)$ tends to the essential supremum[4] of X as $\alpha \uparrow 1$, so we define $\mathrm{CVaR}_1(X) := \operatorname{ess\,sup}(X)$. Let $F_X(x) := \mathrm{Prob}(X \le x)$ be the cumulative distribution function (CDF) of X and

$$q_\alpha^-(X) := \inf\{t : F_X(x) \ge \alpha\},\ \ q_\alpha^+(X) := \sup\{t : F_X(x) \le \alpha\},$$

be the left side and right side quantiles of X. If $q_\alpha^-(X) = q_\alpha^+(X)$ we simply denote it by $q_\alpha(X)$. It is well known that for $\alpha \in (0,\ 1)$ the minimum in the right hand side of (2.1) is attained for any $t \in [q_\alpha^-(X), q_\alpha^+(X)]$.

Denote $\bar{q}_\alpha(X) := \mathrm{CVaR}_\alpha(X)$. For $x \in \mathbb{R}$, consider the equation

$$x = \mathrm{CVaR}_\alpha(X), \tag{2.2}$$

with respect to $\alpha \in [0,\ 1]$. It follows from the representation

$$\mathrm{CVaR}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 q_\tau^-(X)d\tau, \tag{2.3}$$

that $\mathrm{CVaR}_\alpha(X)$ is continuous and monotonically increasing in $\alpha \in [0, 1-\kappa\,]$, where $\kappa := \mathrm{Prob}\{X = \operatorname{ess\,sup}(X)\}$. Hence Eq. (2.2) has unique solution $\alpha = \bar{q}_x^{-1}(X)$ for $\mathbb{E}[X] \le x < \operatorname{ess\,sup}(X)$. The buffered probability of exceedance of a random variable X is defined as

$$\bar{p}_x(X) := \begin{cases} 1 - \bar{q}_x^{-1}(X) & \text{if}\quad \mathbb{E}[X] < x < \operatorname{ess\,sup}(X), \\ 1 & \text{if}\quad x \le \mathbb{E}[X], \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

That is,

$$\mathrm{CVaR}_{1-\bar{p}_x(X)}(X) = x,\ \text{ when } \mathbb{E}[X] \le x < \operatorname{ess\,sup}(X).$$

Consider the following representation of the buffered probability of exceedance of a random variable X (cf. Mafusalov & Uryasev, 2014, Proposition 1):

$$\bar{p}_x(X) = \begin{cases} \inf_{a\ge 0} \mathbb{E}[a(X-x)+1]_+ & \text{if}\quad x < \operatorname{ess\,sup}(X), \\ 0 & \text{if}\quad x \ge \operatorname{ess\,sup}(X). \end{cases} \tag{2.5}$$

Consider

$$\Psi(a, X) := [a(X-x)+1]_+ \ \text{ and }\ \psi(a) := \mathbb{E}[\Psi(a, X)]. \tag{2.6}$$

Note that $\Psi(a,\ X)$ and hence $\psi(a)$ are convex functions of a. For $\mathbb{E}[X] < x < \operatorname{ess\,sup}(X)$ the set of minimizers $\arg\min_{a\ge 0}\psi(a)$ forms a closed interval $[a_1,\ a_2]$, where

$$a_1 = 1/(x - q_\alpha^-(X)) \ \text{ and }\ a_2 = 1/(x - q_\alpha^+(X)), \tag{2.7}$$

with α defined by Eq. (2.2). In particular if the quantile $q_\alpha(X)$ is unique, i.e., $q_\alpha(X) = q_\alpha^-(X) = q_\alpha^+(X)$, then

$$\bar{a} = 1/(x - q_\alpha(X)) \tag{2.8}$$

is the unique minimizer of the right hand side of (2.5).

For $\alpha \in (0,\ 1)$ we have that $\mathrm{CVaR}_\alpha(X) > q_\alpha^-(X)$, and hence the numbers a_1 and a_2 are positive when $\mathbb{E}[X] < x$. When $x < \mathbb{E}[X]$, the minimizer in (2.5) is $\bar{a} = 0$, and $\bar{p}_x(X) = 1$. When $x < \operatorname{ess\,sup}(X)$ we have that $X < x$ w.p.1, and hence $\inf_{a\ge 0}\psi(a) = 0 = \bar{p}_x(X)$. When $x = \operatorname{ess\,sup}(X)$,

$$\inf_{a\ge 0}\psi(a) = \mathrm{Prob}(X = x), \tag{2.9}$$

[3] We use notation $[a]_+ := \max\{0, a\}$ for $a \in \mathbb{R}$.

[4] The essential supremum $\operatorname{ess\,sup}(X)$ can be $+\infty$ if the random variable X is unbounded.