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Decision Support

Optimal asset allocation: Risk and information uncertainty

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ABSTRACT

In asset allocation problem, the distribution of the assets is usually assumed to be known in order to identify the optimal portfolio. In practice, we need to estimate their distribution. The estimations are not necessarily accurate and it is known as the uncertainty problem. Many researches show that most people are uncertainty aversion and this affects their investment strategy. In this article, we consider risk and information uncertainty under a common asset allocation framework. The effects of risk premium and covariance uncertainty are demonstrated by the worst scenario in a set of measures generated by a relative entropy constraint. The nature of the uncertainty and its impacts on the asset allocation are discussed.

1. Introduction

Mean-variance approach introduced by Markowitz (1952) inspires numerous studies in the asset allocation problem. The key idea of mean-variance approach is to consider the optimal portfolio selection as a balance between reward and risk, where they are quantified using expected value and variance (standard deviation) of the portfolio return. This seminal idea attracts a lot of attention and it becomes the foundation of many researches in this area. It plays an important role in the development of some important concepts in finance, such as efficient frontier (Merton, 1972) and the capital asset pricing model (Sharpe, 1964). Researchers investigate and modify the basic mean-variance model and try to obtain more practical results on asset allocation. For example, Li, Chan, and Wan (1998) minimize the probability of a significant loss and study the asset allocation problem in a multi-period model. The multi-period framework is also applied on a mean-variance formulation in Li and Ng (2000). Bertsimas, Lauprete, and Samarov (2004) use a conditional expected loss to replace variance for portfolio optimization. In these researches, the distributions of the assets are assumed to be known by the investors with a full certainty.

Involving uncertainty is another direction of study in asset allocation problem. In the real market, the exact distributions of the risky assets are normally unknown. We can only estimate the distribution of the assets by the historical data and personal experience. The estimation is not necessarily accurate and it is known as the uncertainty (or ambiguity) of the assets' distribution. The famous Ellsberg paradox suggests that people are uncertainty averse. There are many researches about risk. It is interesting and important to know more about the nature of uncertainty and understand its effects on decision making. Various models are suggested to analyse the effect of uncertainty. Different researchers have different focuses on their uncertainty model. Quiggin (1982), Schmeidler (1989) and Tversky and Kahneman (1992) use a non-additive probability setting to model the uncertainty aversion character. One key feature of these models is that the result is consistent with stochastic dominance and so they have great contributions to the development of behavioural finance. Some researches pay more attention to the mathematical and statistical nature of parameter uncertainty. Yaari (1987) constructs a dual theory to demonstrate uncertainty aversion of agents. DeMigual and Nogales (2009) replace mean and variance with more robust reward and risk measures to increase the consistency of the mathematical results. Bodnar, Parolya, and Schmid (2013) consider statistical errors on the parameters. Stochastic models can also be introduced for the uncertain parameters. Gennotte (1986) identifies the optimal investment strategy with a stochastic model on covariance matrix. Klibanoff, Marinacci, and Mukerji (2005) use the idea of utility function to demonstrate uncertainty aversion. Huang and Ying (2013) apply the concept of fuzzy logic to model asset return.

Another way to model uncertainty is to consider a set of parameters (which represents different scenarios or distributions) rather than a point estimation of the parameters. It is sometimes known as the robust optimization. Bertsimas, Brown, and Caramanis (2007) provide more details and mathematical setting of this approach. As

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the actual parameters are unknown, by using a set of parameters, a more robust result can be obtained. This idea is commonly used in engineering, operations research, financial economics and many other subjects. There are different ways to generate a set of parameters. Tütüncü and Koenig (2004) and Epstein and Schneider (2008) consider intervals which are likely to include the actual parameters. Gregory, Darby-Dowman, and Mitra (2011) further introduce a boundary for the number of uncertainty parameters. Hansen and Sargent (2001) and Lim and Shanthikumar (2007) use the concept of relative entropy to generate the set of parameters. Scutellà and Recchia (2013) propose more methods for generating a range of parameters.

There are various statistical methods to estimate the value of unknown parameters. The true value of the parameters is expected to be around the estimator. When a set of probability measures is used to model uncertainty, it should be consistent with the statistical results and not too far away from the estimation. Relative entropy measures the difference between distributions (or probability measures). Hence, it can reasonably be used to generate an uncertainty set of parameters. It is done by setting a constraint on the deviation between the best estimated measure and uncertainty measures. The worst scenario in the uncertainty set is then chosen to illustrate the uncertainty aversion of the investors. Relative entropy has been used in various optimization problems, including asset allocation. Hansen and Sargent (2001) use relative entropy to model uncertainty and obtain the deterministic optimal investment strategy. Calafiore (2007) considers a discrete number of scenarios and studies the computational algorithm of solving the problem. Yuen and Yang (2012) replace variance by expected loss to measure investment risk and identify the relationship between risk and uncertainty. Existing researches mainly focus on the uncertainty of risk premia and its impacts on the investment strategy. Here, we introduce a model on the uncertainty of the interaction between the returns of different assets in order to study its mathematical and financial properties in asset allocation problem.

In this article, we study the asset allocation problem with uncertainty using the mean-variance approach. We apply the multivariate normal distribution to model the returns of the assets. People are assumed to be uncertainty averse. They are conservative when they make a decision in an uncertain situation. Relative entropy is used to generate a set of probability measures which demonstrates the uncertainty of the parameters. The set is found to have some nice mathematical properties which are important in modelling covariance uncertainty. Different measures in the set refer to different scenarios in the market. The worst scenario is used to study the uncertainty aversion characteristic of the agents. Through this model construction, we can obtain the properties of the worst scenario and the corresponding optimal investment strategy. In the following, we focus on expected return uncertainty in Section 2 and covariance uncertainty in Section 3. In Section 4, we present more mathematical details of the model and consider the two sources of uncertainty together. In Section 5, numerical examples are used to illustrate the ideas of our model. More characteristics of the model are also discussed.

2. Model formulation and uncertainty on risk premia

We assume that there are n risky assets in the market. Their returns follow multivariate normal distribution. Let r_f be the risk-free rate, μ and V be the risk premium and the covariance matrix of the risky assets' returns under the physical probability measure P, respectively. Here, P represents the best estimated market environment. Hence, μ and V are the best estimations towards the parameters of returns of these assets by the investors using all available information, including historical data, news, their knowledge, etc. Under mean-variance approach, they are linked to the reward and the risk of investment. We also assume that there are no redundant risky assets in the market. Hence, V is symmetric and positive definite. For square matrices M and N with the same dimension, we write $M \succ N$ ($M \succeq N$), if, $M - N$ is positive definite (semi-definite). We have $V \succ 0$.

It is possible for the estimated parameters, μ and V, to have large differences with the actual parameters (uncertainty). That means, the expected investment performance and the inter-relationship of these assets are different from our estimation. The optimal portfolio induced by these two parameters can be inappropriate. Apart from considering the risk of the assets' return, we also need to study the effects of uncertainty of the model parameters. The investors might consider a range of scenarios based on their estimations to identify the potential loss if the market deviates from their prediction. In our model, $\mathcal{Q}$ denotes the set of uncertainty measures representing these scenarios. It illustrates the effects of uncertainty in the decision making process. Let $p(x)$ and $q(x)$ be the probability density functions of the risky assets' returns under measures P and $Q \in \mathcal{Q}$, respectively. If there is no uncertainty on the parameters, measure P is used directly to study the problem.

We can now construct the set $\mathcal{Q}$ using the idea that the measures in $\mathcal{Q}$ should not greatly deviate from P. Relative entropy is used to measure the deviation between two probability measures. We assume that the relative entropy of all measures in the uncertainty set $\mathcal{Q}$ with respect to P is not greater than a positive constant K. That is,

$$KL(Q, P) := \int q(x) \ln \frac{q(x)}{p(x)} dx \le K, \qquad \forall Q \in \mathcal{Q}. \tag{1}$$

The parameter K depends on the investors' confidence on the available information and their opinions on the market when they make an investment decision. It is greater when they are more conservative in investing these risky assets. The uncertainty set need not be large enough to cover all possible (extreme) scenarios that the investors can imagine. However, it can cover the adverse scenarios that the investors think important when they make the decision. In scenario $Q \in \mathcal{Q}$, the risk premium and the return covariance of the risky assets change and they are denoted by $\hat{\mu}$ and $\hat{V}$, respectively. For a square matrix M, let $\mathrm{tr}(M)$, $|M|$ and M' be the trace, the determinant and the transpose of M, respectively. We can find the explicit form of $KL(Q, P)$ with the following equation,

$$\begin{aligned} KL(Q, P) = \frac{1}{2}\big[&\ln|V| - \ln|\hat{V}| + \mathrm{tr}(V^{-1}\hat{V}) - n \\ &+ (\mu - \hat{\mu})' V^{-1} (\mu - \hat{\mu})\big]. \end{aligned} \tag{2}$$

We now apply the uncertainty model on our asset allocation problem. The investors realize that the distribution of the return of the assets can be different from their expectation due to various reasons (estimation errors, distribution changing over time and etc.). Due to their uncertainty aversion, they are more conservative in the real market compared with a market with no uncertainty. The worst scenario in the set of measures is chosen in our analysis. It can illustrate the uncertainty aversion behaviour of the investors. Let $\mathcal{U}$ be the set of parameters $(\hat{\mu}, \hat{V})$ under the measure $Q \in \mathcal{Q}$. In this section, we assume that V is fixed in the set $\mathcal{U}$ and use $\mathcal{U}_\mu$ to denote the corresponding feasible set of parameters. Using the mean-variance approach, if r_p is the required risk premium of the portfolio in the worst scenario which is non-negative, u is the composition of the portfolio, the asset allocation problem can be formulated as

$$\begin{aligned} &\min_{u \in \mathbf{R}^n} \quad u'Vu \\ &\text{such that} \quad u'\hat{\mu} \ge r_p \text{ for all } (\hat{\mu}, V) \in \mathcal{U}_\mu \text{ and } KL(Q, P) \le K. \end{aligned} \tag{3}$$

As V is fixed, $\hat{\mu}$ is the only source of uncertainty. The relative entropy function can be simplified as

$$KL(Q, P) = \frac{1}{2}\big[(\mu - \hat{\mu})' V^{-1} (\mu - \hat{\mu})\big].$$