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Original Article

# Complexity of graphs generated by wheel graph and their asymptotic limits

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ABSTRACT

The literature is very rich with works deal with the enumerating the spanning trees in any graph $G$ since the pioneer Kirchhoff (1847). Generally, the number of spanning trees in a graph can be acquired by directly calculating an associated determinant corresponding to the graph. However, for a large graph, evaluating the pertinent determinant is ungovernable. In this paper, we introduce a new technique for calculating the number of spanning trees which avoids the strenuous computation of the determinant for calculating the number of spanning trees. Using this technique, we can obtain the number of spanning trees of any graph generated by the wheel graph. Finally, we give the numerical result of asymptotic growth constant of the spanning trees of studied graphs.



## 1. Introduction

The research of the complexity of a graph has a comparatively long history. The importance of this research line is in fact due to:

1- Investigating the possible particle transitions of masers using energy analysis,
2- Estimating the accuracy of a network,
3- Recounting specific chemical isomers,
4- Electrical circuits layout,
5- Enumerating the number of Eulerian tours in a graph [1–10].

The complexity (the number of spanning trees) $\tau(G)$ of a finite connected undirected graph $G$ is defined as the total number of distinct connected acyclic spanning subgraphs.

There are many techniques to compute this number. Kirchhoff [11] gave the famous matrix tree theorem. In which $\tau(G) =$ any cofactor of $L(G)$, where $L(G)$ is equal to the degree matrix $D(G)$ of $G$ minus the adjacency matrix $A(G)$ of $G$.

Another method to count the complexity of a graph is using Laplacian eigenvalues. Let $G$ be a connected graph with $n$ vertices. Kelmans and Chelnoknov [12] derived the following formula:

$$\tau(G) = \frac{1}{n}\prod_{k=1}^{n-1}\mu_k. \tag{1}$$

Where $n = \mu_1 \geq \mu_2 \geq \dots \geq \mu_n = 0$ are the eigenvalues of the Laplacian matrix $L(G)$.

Degenerating the graph through successive elimination of contraction of its edges represent the core of another way to compute the complexity of a graph [13]. In this way, the summation of complexities in small well known graphs yields directly the complexity of an unknown graph $G$. Let $e$ be an edge with endpoints $u$ and $v$ in the graph $G$, the deletion $G-e$ of $e$ from $G$ is the graph gained by removing $e$ and the contraction $G\cdot e$ of $e$ from $G$ is the graph obtained by removing $e$ and identifying $u$ and $v$. The formula for computing the complexity of a graph $G$ is given by

$$\tau(G) = \tau(G-e) + \tau(G\cdot e). \tag{2}$$

Recently, Daoud [14] introduced some new theorems which generalized this method. We will make use of these theorems in this work.

## 2. Main results

**Theorem 1.** For $n \geq 3$, the number of spanning trees of the wheel graph $W_n$ is given by $\left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2$.

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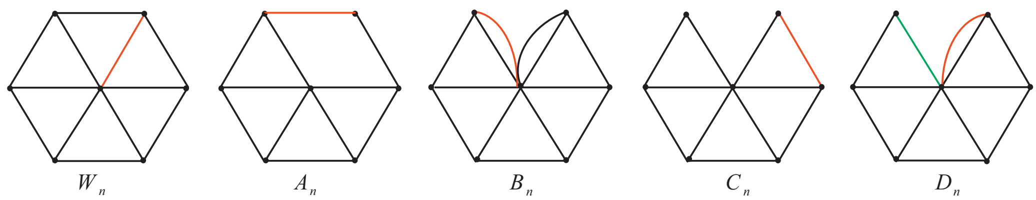


**Fig. 1.** The five families of graphs which we use to find an explicit formula for the complexity in the wheel graph $W_n$.

**Proof:** Consider the following five different families of graphs denoted by $W_n,\ A_n,\ B_n,\ C_n$ and $D_n$ as shown in Fig. 1, where $n$ denote the number of vertices.

We use Eq. (2) on the indicated edges to find a system of recurrence relations:

$$\tau(W_n) = \tau(A_n) + \tau(B_{n-1})$$
$$\tau(A_n) = \tau(C_{n-1}) + \tau(W_{n-1})$$
$$\tau(B_n) = \tau(D_n) + \tau(B_{n-1})$$
$$\tau(C_n) = \tau(C_{n-1}) + \tau(D_{n-1})$$
$$\tau(D_n) = \tau(C_n) + \tau(D_{n-1}) = \tau(D_{n-1}) + \tau(B_{n-1}).$$

Consider the last two relations for $\tau(C_n)$ and $\tau(D_n)$, we have: $\tau(C_{n+1}) = \tau(C_n) + \tau(D_n) = 2\,\tau(C_n) + \tau(D_{n-1}) = 3\,\tau(C_n) - \tau(C_{n-1})$, or $\tau(C_{n+1}) - 3\,\tau(C_n) + \tau(C_{n-1}) = 0$, thus $\tau(C_n) - 3\,\tau(C_{n-1}) + \tau(C_{n-2}) = 0$ and $\tau(C_{n-1}) - 3\,\tau(C_{n-2}) + \tau(C_{n-3}) = 0$.

Subtracting these two equations, we get $\tau(C_n) - 4\,\tau(C_{n-1}) + 4\,\tau(C_{n-2}) - \tau(C_{n-3}) = 0$, which is the final recurrence relation for $\tau(C_n)$.

Consider the first two relations for $\tau(W_n)$ and $\tau(A_n)$.

Since $\tau(B_{n-1}) = \tau(C_n)$, we have $\tau(W_n) = \tau(A_n) + \tau(C_n)$ and hence $\tau(W_{n-1}) = \tau(A_{n-1}) + \tau(C_{n-1})$.

Substituting into the second relation, we obtain $\tau(A_n) = \tau(A_{n-1}) + 2\,\tau(C_{n-1})$, therefore $\tau(A_n) - \tau(A_{n-1}) = 2\,\tau(C_{n-1})$, since $\tau(C_{n-1}) - 3\,\tau(C_{n-2}) + \tau(C_{n-3}) = 0$, we have $2\,\tau(C_{n-1}) - 2\,(3)\,\tau(C_{n-2}) + 2\,\tau(C_{n-3}) = 0$, $[\tau(A_n) - \tau(A_{n-1})] - 3\,[\tau(A_{n-1}) - \tau(A_{n-2})] + [\tau(A_{n-2}) - \tau(A_{n-3})] = 0$, thus $\tau(A_n) - 4\,\tau(A_{n-1}) + 4\,\tau(A_{n-2}) - \tau(A_{n-3}) = 0$, which is the final recurrence relation for $\tau(A_n)$.

Now both $\tau(A_n)$ and $\tau(C_n)$ have the third order homogeneous recurrence relation:

$$x_n - 4\,x_{n-1} + 4\,x_{n-2} - x_{n-3} = 0. \tag{3}$$

Thus $\tau(W_n) = \tau(A_n) + \tau(C_n)$ must have the same relation. Therefore the characteristic equation corresponding to this recurrence relation is $r^3 - 4r^2 + 4r - 1 = 0$, which has characteristic roots $r = \frac{3\pm\sqrt{5}}{2}$ and $r = 1$. Therefore, the general solution of $\tau(W_n)$ is $\tau(W_n) = \alpha(\frac{3+\sqrt{5}}{2})^n + \beta(\frac{3-\sqrt{5}}{2})^n + \gamma$.

Solution of the recurrence relation (3) now reduces to find the values of the constants $\alpha,\ \beta$ and $\gamma$ such that the general solution conforms with the given initial conditions $\tau(W_3) = 16,\ \ \tau(W_4) = 45$ and $\tau(W_5) = 121$. Substituting the initial conditions in the general solution we obtain

$$\tau(W_n) = \alpha\left(\tfrac{3+\sqrt{5}}{2}\right)^3 + \beta\left(\tfrac{3-\sqrt{5}}{2}\right)^3 + \gamma = 16$$
$$\tau(W_n) = \alpha\left(\tfrac{3+\sqrt{5}}{2}\right)^4 + \beta\left(\tfrac{3-\sqrt{5}}{2}\right)^4 + \gamma = 45$$
$$\tau(W_n) = \alpha\left(\tfrac{3+\sqrt{5}}{2}\right)^5 + \beta\left(\tfrac{3-\sqrt{5}}{2}\right)^5 + \gamma = 121.$$

This system of equations have a unique solution $\alpha = \beta = 1$ and $\gamma = -2$, and hence the result follows. □

The gear graph $G_n$, is the graph obtained from $W_n$ by inserting a vertex between any two adjacent vertices in its cycle $C_n$. See Fig. 2.

**Theorem 2.** For $n \geq 3$, the number of spanning trees of the gear graph $G_n$ is given by $(2+\sqrt{3})^n + (2-\sqrt{3})^n - 2$.

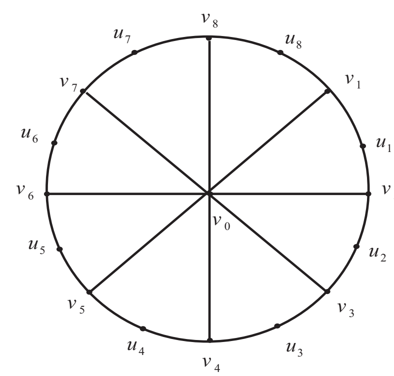


**Fig. 2.** The gear graph $G_8$.

**Proof:** Consider the following five different families of graphs denoted by $G_n,\ A_n,\ B_n,\ C_n$ and $D_n$ as shown in Fig. 3, where $n$ denote the number of vertices of $W_n$.

We use Eq. (2) together with Theorem 2.2 in [14] on the indicated edges and paths to find a system of recurrence relations:

$$\tau(G_n) = \tau(A_n) + \tau(B_{n-1})$$
$$\tau(A_n) = 2\,\tau(C_{n-1}) + \tau(G_{n-1})$$
$$\tau(B_n) = 2\,\tau(D_n) + \tau(B_{n-1})$$
$$\tau(C_n) = \tau(C_{n-1}) + \tau(D_{n-1})$$
$$\tau(D_n) = 2\,\tau(C_n) + \tau(D_{n-1}) = \tau(D_{n-1}) + \tau(B_{n-1}).$$

Consider the last two relations for $\tau(C_n)$ and $\tau(D_n)$, we have $\tau(C_{n+1}) = \tau(C_n) + \tau(D_n) = 3\ \tau(C_n) + \tau(D_{n-1}) = 4\,\tau(C_n) - \tau(C_{n-1})$, or $\tau(C_{n+1}) - 4\,\tau(C_n) + \tau(C_{n-1}) = 0$, Thus $\tau(C_n) - 4\,\tau(C_{n-1}) + \tau(C_{n-2}) = 0$ and $\tau(C_{n-1}) - 4\,\tau(C_{n-2}) + \tau(C_{n-3}) = 0$.

Subtracting these two equations, we get $\tau(C_n) - 5\,\tau(C_{n-1}) + 5\,\tau(C_{n-2}) - \tau(C_{n-3}) = 0$, which is the final recurrence relation for $\tau(C_n)$. Consider the first two relations for $\tau(G_n)$ and $\tau(A_n)$.

Since $\tau(B_{n-1}) = 2\,\tau(C_n)$, we have $\tau(G_n) = \tau(A_n) + 2\,\tau(C_n)$ and hence $\tau(G_{n-1}) = \tau(A_{n-1}) + 2\,\tau(C_{n-1})$.

Substituting into the second relation, we obtain $\tau(A_n) = \tau(A_{n-1}) + 4\,\tau(C_{n-1})$, therefore $\tau(A_n) - \tau(A_{n-1}) = 4\,\tau(C_{n-1})$, since $\tau(C_{n-1}) - 4\,\tau(C_{n-2}) + \tau(C_{n-3}) = 0$, we have $4\,\tau(C_{n-1}) - 4\,(4)\,\tau(C_{n-2}) + 4\,\tau(C_{n-3}) = 0$, $[\tau(A_n) - \tau(A_{n-1})] - 4\,[\tau(A_{n-1}) - \tau(A_{n-2})] + [\tau(A_{n-2}) - \tau(A_{n-3})] = 0$, thus $\tau(A_n) - 5\,\tau(A_{n-1}) + 5\,\tau(A_{n-2}) - \tau(A_{n-3}) = 0$, which is the final recurrence relation for $\tau(A_n)$.

Now both $\tau(A_n)$ and $\tau(C_n)$ have the third order homogeneous recurrence relation:

$$x_n - 5\,x_{n-1} + 5\,x_{n-2} - x_{n-3} = 0. \tag{4}$$

Thus the characteristic equation corresponding to this recurrence relation is $r^3 - 5\,r^2 + 5\,r - 1 = 0$, which has characteristic roots $r = 2 \pm \sqrt{3}$ and $r = 1$. Thus the general solution of $\tau(G_n)$ is $\tau(G_n) = \alpha\,(2+\sqrt{3})^n + \beta\,(2-\sqrt{3})^n + \gamma$.

Solution of the recurrence relation (4) now reduces to find the values of the constants $\alpha,\ \beta$ and $\gamma$ such that the general solution conforms with the given initial conditions $\tau(G_3) = 50\,,\ \ \tau(G_4) = 192$ and $\tau(G_5) = 722$. Substituting the initial conditions in the gen-