Journal of the Egyptian Mathematical Society 25 (2017) 167–170

Contents lists available at ScienceDirect

Journal of the Egyptian Mathematical Society

journal homepage: www.elsevier.com/locate/joems

Original article

New types of winning strategies via compact spaces

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ARTICLE INFO

Article history:
Received 21 October 2016
Accepted 29 December 2016
Available online 26 January 2017

MSC:
54B05
54B10
54C10
54D18
90D42

Keywords:
$\Gamma^*(T_i)$-compactness
$\Gamma^*(T_j)$-compactness
$\Pi^*(T_i)$-compactness
$\Pi^*(T_j)$-compactness

ABSTRACT

Current paper aims to introduce new types of compactness in terms of notion of $\mathcal{K}$-cover in topological games with perfect information of Telgársky,namely, $\Gamma^*(T_i)$-compactness, $\Gamma^*(T_j)$-compactness, $\Pi^*(T_i)$-compactness and $\Pi^*(T_j)$-compactness in the realm of Hausdörff spaces. We give a necessary and sufficient condition for players to have a winning strategy in these types of compactness. Furthermore, various characterizations of these concepts are achieved.

1. Introduction and preliminaries

Compactness in game theory plays an essential role when general topology was developed. Many authors defined and studied some types of compactness through conceps of game theory.

Berge [1] has introduced and studied the notion of topological games with perfect information. The concept of topological games $G(\mathcal{K}, X)$ was introduced and studied by Telgársky [2]. He defined and investigated spaces through topological games as C-scattered and paracompact spaces [3], compact-like spaces [4]. Galvin et al. ([5,6])introduced some stationary strategies in topological games. They studied infinite games in [7]. Junnila et al. [8] studied closure-preserving covers by small sets. Banakh and Zdomskyy [9] introduced and studied some separation properties say $\mathcal{C}$-separation properties between the σ-compactness and Hurewicz property. Tkachuk in [10] discussed Eberlein compact and weakly Eberlein compact spaces from the view of functional analysis and C_p-theory. Paulo Klinger Monteiro and Frank H. Page Jr [11] introduced a condition, uniform payoff security, for games with compact Hausdorff strategy spaces and payoffs bounded and measurable in players strategies. Bennett, Lutzer and Reedc [12] proved a Moore space the equivalence between domain representability; subcompactness; the existence of a winning strategy for player α(= the nonempty player) in the strong Choquet game $Ch(X)$; the existence of a stationary winning strategy for player α in $Ch(X)$; and Rudin completeness. Scheepers and Tsaban [13] extended studies of selection principles for families of open covers of sets of real numbers to include families of countable Borel covers. They proved that some of the classes which were different for open covers are equal for Borel covers. Cao et al. [14] studied some two person games and some topological properties defined by them. Zoroa et al. [15] studied games in which the strategic situation is developed on a lattice. The main characteristic of these games is that the points in each column of the lattice have a specific associated weight which directly affects the payoff function.

In this paper, we introduce and study new types of compactness say $\Gamma^*(T_i)$-compactness, $\Gamma^*(T_j)$-compactness, $\Pi^*(T_i)$-compactness and $\Pi^*(T_j)$-compactness in the realm of Hausdorff spaces. The paper based on an infinite topological game. In this game players I and II alternately choose points and their open neighborhoods respectively. I wins if and only if the moves of II cover the space. All spaces are assumed to be T_1. In particular, compact spaces and paracompact spaces are assumed to be Hausdorff or T_2.

2. Some basic definitions

A topological space [16] is a pair (X, τ) consisting of a set X and family τ of subsets of X satisfying $X, \phi \in \tau$, τ is closed under arbitrary union and closed under finite intersection. Each member in τ is said to be an open set. The complement of each open set

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http://dx.doi.org/10.1016/j.joems.2016.12.003
1110-256X/

is a closed set. 2^X will be denote to the class of all closed sets in a space X. $\mathcal{K}$ will be denote to a class of topological spaces which are hereditary with respect to closed sets. The letters i, j, and k denote nonnegative integers. A topological space (X, τ) is said to be compact [16] if each open cover of X has a finite subcover. A Lidelöff space is a topological space in which every open cover has a countable subcover. A Lidelöff space is a weakening of compactness, which requires the existence of a finite subcover. Telgársky [2] introduced the definition of $\mathcal{K}$-cover of a space X. A family $\mathcal{A}$ of open subsets of X is a $\mathcal{K}$-cover of X if for each $E \in 2^X \cap \mathcal{K}$, there exists $A \in \mathcal{A}$ such that $E \subseteq A$.

Definition 2.1 [3]. A strategy s for player I is a function whose domain is the set of finite sequences of nonempty open sets U_i of a space X and s has the property that if $\langle U_1, U_2, \cdots, U_k\rangle$ is a finite sequence, then $s\ \langle U_1, U_2, \cdots, U_k\rangle$ is a subset of X. Such strategy for player I is a winning strategy if each play $\langle x_1, U_1, x_2, U_2, \cdots\rangle$ of a game $G(X, \tau)$ for which $x_k = s(\langle U_1, U_2, \cdots, U_k\rangle)$ for each positive integer k is won by player I.

Definition 2.2 [2]. A topological space (X, τ) is called $\mathcal{K}$-compact if each $\mathcal{K}$-cover of X contains a countable cover of X.

Lemma 2.3 [2]. *If Player I has a winning strategy in an infinite positional game $G(\mathcal{K}, X)$, then X is $\mathcal{K}$-compact.*

Lemma 2.4 [2]. *If a topological space (X, τ) is not $\mathcal{K}$-compact, then Player II has a winning strategy in $G(\mathcal{K}, X)$.*

Definition 2.5 [16]. A topological space (X, τ) is called:

(i) A T_1 if for each $x, y \in X$, $x \neq y$, there exist two disjoint open sets U and V such that $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$.

(ii) A Hausdorff or T_2 for each $x, y \in X$, $x \neq y$, there exist two disjoint open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

3. $\Gamma^*(T_i)$(resp. $\Gamma^*(T_j^*)$)-compact spaces

In [17], Aull introduced the notion of α-paracompact subset. A subset E of a space (X, τ) is called α-paracompact in X if every covering of E by open subsets of X has a refinement by open subsets of X which is locally finite in X and covers E.

Lupiáñez [18] used this concept to define the classes $\Gamma^*(T_i)$ and $\Gamma^*(T_j^*)$ for $i = 2, 3, 3a, 4, 5, 5a$ (resp. $j = 4, 5, 5a$). $\Gamma^*(T_i)$ (resp. $\Gamma^*(T_j^*)$) the class of all T_i spaces (resp. T_j^* spaces) which are α-paracompact in each T_i-space (resp. T_j^*-space) in which are embedded as closed subsets.

Definition 3.1. A family $\mathcal{A}$ of open subsets in a space (X, τ) is called $\Gamma^*(T_i)$-cover (resp. $\Gamma^*(T_j^*)$-cover) of X if each $E \in 2^X \cap \Gamma^*(T_i)$ (resp. $E \in 2^X \cap \Gamma^*(T_j^*)$, there exists $A(E) \in \mathcal{A}$ for which $E \subset A(E)$, on the other hand, $2^X \cap \Gamma^*(T_i)$ (resp. $2^X \cap \Gamma^*(T_j^*)$) is a refinement of $\mathcal{A}$.

Definition 3.2. A topological space (X, τ) is said to be $\Gamma^*(T_i)$-compact (resp. $\Gamma^*(T_j^*)$-compact) if each $\Gamma^*(T_i)$-cover (resp. $\Gamma^*(T_j^*)$-cover) of X contains a countable subcover of X.

Theorem 3.3. *If $\mathcal{K}$ is the class of all one-point spaces and the empty space. Then $\mathcal{K}$-compact spaces and Lindelöf spaces coincide.*

Proof. It suffices to show that 1-cover and open cover coincide. Let $\mathcal{K} = \{\{x\} : x \in X\}$ and $\mathcal{A}$ be a $\mathcal{K}$-cover of X. By Definition 2.2, we may assume A is countable. Now for each $x \in X$ we have $\{x\}$. Thus, there exist $A \in \mathcal{A}$ such that $\{x\} \subseteq A$. Then $\bigcup\{\{x\} : x \in X\} \subseteq \bigcup\{A : A \in \mathcal{A}\}$. Hence $X = \bigcup\{A : A \in \mathcal{A}\}$. This proves that $\mathcal{A}$ is open cover. □

Theorem 3.4. *If player I has a winning strategy of an infinite positional game $G(\Gamma^*(T_i), X)$ (resp. $G(\Gamma^*(T_j^*), X)$), then X is $\Gamma^*(T_i)$-compact (resp. $\Gamma^*(T_j^*)$-compact).*

Proof. Let s be a winning strategy of player I and $\mathcal{A}$ be $\Gamma^*(T_i)$-cover of X. For each $E \in 2^X \cap \Gamma^*(T_i)$, there exists $A(E) \in \mathcal{A}$ for which $E \subset A(E)$. We define a strategy t for player II as follows: We set $t(E_0, E_1, \cdots, E_{2n+1}) = \bigcap\{X - A_k(E_{2k+1}) : k \leq n\}$ for each admissible sequence $(E_0, E_1, \cdots, E_{2n+1})$ for $G(\Gamma^*(T_i), X)$. Let $< E_n : n \in \mathbb{N} >$ be a play of $G(\Gamma^*(T_i), X)$, where $E_{2n+1} = s(E_0, E_1, \cdots, E_{2n})$ and $E_{2n+2} = t(E_0, E_1, \cdots, E_{2n+1})$ for each $n \in \mathbb{N}$. Since s is a winning strategy for player I in $G(\Gamma^*(T_i), X)$, then $\bigcup\{E_{2n} : n \in \mathbb{N}\} = X$ and so $\bigcap\{X - A_n(E_{2n+1}) : n \in \mathbb{N}\} = \phi$. Hence $\bigcup\{A_n(E_{2n+1}) : n \in \mathbb{N}\} = X$. □

Lemma 3.5 [18]. *In a topological space (X, τ), the following hold:*

(i) *If X is a Lindelöf T_3 space, then $X \in \Gamma^*(T_4)$.*
(ii) *$\Gamma^*(T_4)$ is the class of Lindelöf T_3 spaces.*

Definition 3.6. Let m be an infinite cardinal. A space X is called m-Lindelöf T_3 if each open cover of X contains a subcover of cardinality $\leq m$.

Theorem 3.7. *For a regular space X; if player I has a winning strategy in $G(\Gamma^*(T_4), X)$ and each $E \in 2^X \cap \Gamma^*(T_4)$ is m-Lindelöf T_3 space, then $X \in \Gamma^*(T_4)$.*

Proof. Let X be a regular space. By Lemma 3.5, it suffices to prove that X is a Lindelöf T_3 space. Let $\mathcal{A}$ be an open cover of X and $\mathcal{B}$ be the family of all $B \subseteq X$ such that for each $B \in \mathcal{B}$, there exists $\{A_i : i \in I\} \subseteq \mathcal{A}$ with $\operatorname{card} I \leq m$ and $\bigcup\{A_i : i \in I\} = B$. Assume that $E \in 2^X \cap \Gamma^*(T_4)$ is m-Lindelöf T_3 space, this means each open cover $\mathcal{A}^*$ of E by open sets of X, there exists a subcover $\{A_j^* : j \in J\} \subseteq \mathcal{A}^*$ with $\operatorname{card} J \leq m$ and $E \subseteq \bigcup\{A_j^* : j \in J\}$. Since $\mathcal{A}^* \subseteq \mathcal{A}$ and by $\mathcal{B}$, then there exists $B \in \mathcal{B}$ such that $\bigcup\{A_j^* : j \in I\} = B$ and so $E \subseteq B$. Hence $\mathcal{B}$ is $\Gamma^*(T_4)$-cover of X. Assume that player I has a winning strategy in $G(\Gamma^*(T_4), X)$. By Theorem 3.4, for $i = 4$, X is $\Gamma^*(T_4)$-compact. Then $\mathcal{B}$ has a countable cover $\{B_n : n \in \mathbb{N}\}$ of X and $X = \bigcup\{B_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$ and $B_n \in \mathcal{B}$, there exists $\{A_i : i \in I\} \subseteq \mathcal{A}$ with $\operatorname{card} I_n \leq m$ and $\bigcup\{A_i : i \in I_n\} = B_n$. Hence $\bigcup\{A_i : i \in I_n, n \in \mathbb{N}\} = \bigcup\{B_n : n \in \mathbb{N}\} = X$. Therefore $\{A_i : i \in I_n, n \in \mathbb{N}\}$ is a subcover of $\mathcal{A}$ with cardinality $\leq m$ and also covers X. This proves that X is m-Lindelöf T_3 space. □

Definition 3.8. For topological spaces X and Y, a map f: $X \to Y$ is perfect if $f(E) \in 2^X$ for each $E \in 2^X$ and if $f^{-1}(y) \in \mathcal{C}$ for each $y \in Y$ where $\mathcal{C}$ is the class of all compact spaces.

Example 3.9. Let $(\mathbb{R}, T)$ be the Michael line,

$$j_1 : \mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q}) \longrightarrow \mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q}) + (\mathbb{R}, T) \times (\mathbb{R} \setminus \mathbb{Q})$$

$$j_2 : (\mathbb{R}, T) \times (\mathbb{R} \setminus \mathbb{Q}) \longrightarrow \mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q}) + (\mathbb{R}, T) \times (\mathbb{R} \setminus \mathbb{Q})$$

Then the mapping onto

$$f : \mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q}) + (\mathbb{R}, T) \times (\mathbb{R} \setminus \mathbb{Q}) \longrightarrow (\mathbb{R}, T) \times (\mathbb{R} \setminus \mathbb{Q})$$

such that $f(j_1(x, y)) = (x, y)$ if $(x, y) \in \mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q}) f(j_2(x, y)) = (x, y)$ if $(x, y) \in \mathbb{R} \times (\mathbb{R} \setminus \mathbb{Q})$ is a perfect mapping.

Lemma 3.10 [18]. *If $X \in \Gamma^*(T_4)$ and Y is a closed subset of X, then $Y \in \Gamma^*(T_4)$.*

Lemma 3.10 can be rewritten as follows: if $X \in \Gamma^*(T_4)$, then $2^X \subseteq \Gamma^*(T_4)$

Definition 3.11. A class $\Gamma^*(T_4)$ is said to be perfect if there exists a perfect mapping $f : X \longrightarrow Y$ such that if $X \in \Gamma^*(T_4)$, then $Y \in \Gamma^*(T_4)$.

From Definitions 3.8, 3.11 and Lemma 3.10, we have the following result.

Theorem 3.12. *Let $\Gamma^*(T_4)$ be a perfect class and there exists a perfect map from X onto Y. Then*