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# Stochastic approximation with series of delayed observations

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The stochastic approximation procedure with series of delayed observations is investigated. The procedure is formed by modifying the Robbins–Monro stochastic approximation procedure to be applicable in the presence of series of delayed observations. The modified procedure depends on a new base concerning the relation between service time of the series and service times of its components. Two loss systems are introduced for application to the proposed procedure. This new situation can be applied to increase the production of items in many fields such as biological, medical, life time experiments, and some industrial projects, where items are realized after random time delays. The efficiency of the procedure is computed. Our proposal is general and we expect that it can be applied to any other stochastic approximation procedure.



## 1. Introduction

Stochastic approximation and related topics, as recursive parameter estimation, or up and down experimentation, belong to the so-called on-line methods, where the solution of the problem takes place in real time, in the course of the process of observation. Stochastic approximation is a procedure for finding the root of an equation, or the solution of a system of equations, where the values of the respective functions can only be observed (measured) with experimental errors, at recursively determined points. The procedure is nonparametric with respect to the type of the function as well as with respect to the distribution of the experimental error (if an information about this distribution is available, it can be made use of).

The procedure for finding the root is called the Robbins–Monro stochastic approximation procedure [1]. There is an extensive literature and a lot of papers on this topics (cf. [2–7]), we shall use the review papers [8–13] as references.

Typically, in the investigated stochastic approximation procedure, observations follow each other after fixed time-intervals; where the point of the next observation is corrected according to the result of the preceding one. However, in some situations, as in biological or lifetime experiments, it may happen that the result of an observation becomes known only after a random time delay.

Recently, the stochastic approximation has been used in clinical applications to find optimal dose as in [14].

It is meaningful to ask, whether and how stochastic approximation can then be applied. We answer the question for the modified stochastic approximation procedure with delayed series of delayed multiservice observations (or customers) by investigating two loss systems to be applicable in the presence of the modified Robbins–Monro stochastic approximation procedure. A series of delayed observations arrives to the system each time unit where the service time is an integer-valued random variable. Servers are parallel, and there is no waiting places if all servers are busy. According to this new approach, a server of one of the two loss systems cans serve a series of delayed multiservice observations. The number of served observations will be increased and the number of lost observations will be minimized. This approach is not applied in the papers [8,10–13]. In these papers the problem was discussed by applying special loss systems where servers cannot receive (serve) any observation during the time between any two consecutive arrivals.

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However, in the proposed procedure, we introduced a new situation by investigating two loss systems, where the server cans receive an observation or more during the time between any two consecutive series of observations. In fact, the investigation of the mentioned two loss systems was inspired exactly by this application, where it will minimize the number of lost observations. It is proved that the service time of a series equals the sum of service times of its components (observations) where our procedure depends exactly on this new base. The service time of a series is independent of the number of its components and this can be used to increase the number of served observations of the series.

The service time distribution of the loss system, whose arrivals are series of observations, depends on the service time distribution of the loss system whose arrivals are multiservice observations. The probability service time of a series equals the sum of products of probabilities service times of observations without delay terminated by with delay. If the service times of observations are terminated with delay, then all next observations are lost, but if they are terminated without delay, then there is no loss of any observation. The exact bounded service time distribution of a series of delayed observations is obtained by approximating the unbounded geometric random time delay distribution based on the procedure [8], by the bounded service time distribution based on the modified procedure. The approximation depends on the number of multiservice observations of the series as well as it depends on the efficiency of the modified procedure where the number of observations can be increased. This new approach minimizes the number of lost observations and increases the number of served observations, therefore it can be applied to increase the production of items in many fields where items are realized after random time delays. The efficiency of the proposed procedure and its approximation by the efficiency of the procedure [8] are calculated where the results obtained show that the approximation seems to be acceptable. This proves that our proposal can serve as a model of the Robbins–Monro stochastic approximation procedure.

The proposed procedure is more general than the procedure given by the paper [13]. The results in paper [13] can be obtained in the special case that each series contains one observation only.

The investigated procedure is new and we expect that it can be applied to any stochastic approximation or recursive estimation procedure.

## 2. The modified stochastic approximation procedure with delayed series of delayed multiservice observations for independent random time delay distributions

The stochastic approximation procedure with delayed observations is modified to be applicable in the presence of delayed series of delayed multiservice observations. This application will increase the number of served observations in each series. To obtain the exact service time distribution of the series of delayed observation, the unbounded geometric random time delay distribution based on the procedure [8], is approximated by the bounded service time distribution based on the modified procedure. Two loss systems are investigated where arrivals are series of multiservice observations and a series of delayed observations arrives each time unit, service time is an integer-valued random variable, servers are parallel stations, and there is no waiting places if all servers are busy.

### *2.1. Stochastic approximation procedure with delayed observations*

The Robbins–Monro stochastic approximation procedure with delayed observations has been investigated previously for a geometrical delay distribution [8]. To eliminate or diminish time losses due to delays of observations, it has been proposed that experiments (or observations) are allocated into $K$ parallel series in the following way.

The experiment is based on three essential elements, that is, deterministic arrivals, $K$ parallel series, and no queue. The $K$ series are either open or closed at points $x^{(k)}_{n_k}$, where $1 \le k \le K$, $n_k - 1$ is the number of observations realized in the $k$th series up to time $n - 0$ (i.e. immediately before time $n$). At the beginning, i. e., before time $n = 1$, all series are open, all $n_k$ are equal to 1, and all $x^{(k)}_1$ are equal to the same constant. At time $n$, an experiment is made at point $x^{(i)}_{n_i}$, where $i$ is the open series with the smallest $n_i$ and smallest $i$ among them. The $i$th series is then closed at the same $x^{(i)}_{n_i}$ till time $(n + t(n) + 1) - 0$ when it opens at the point

$$x^{(i)}_{n_i+1} = x^{(i)}_{n_i} - a_{n_i}\left(r\left(x^{(i)}_{n_i}\right) + e\left(n + t(n) + 1, x^{(i)}_{n_i}\right)\right).$$

Here $r(x)$ is a function whose zero point $\theta$ is to be found; $e(\upsilon + 1, x)$ is the observational error (noise) corresponding to an observation of $r$ made at point $x$, which becomes known during the interval $[\upsilon,\ \upsilon + 1]$; $x^{(i)}_{n_i}$ is the current approximation to $\theta$ in the $i$th series at time $n - 0$; $a_n, n \in N$, is a zero sequence of positive constants, typically $a_n = \frac{a}{n+n_0}$, where $n_0$ is non-negative; and $t(n)$ is [the integer part of] the delay of the result of an experiment made at time $n$.

If there is no open series at time $n - 0$, no experiment is made at time $n$ and a time-loss is thus incurred. If $l$ is the steady state probability of such a time loss, its complement, $e = 1 - l$, is called the efficiency of the procedure.

To find $\theta$, the average of the current approximations over all series, $\theta_n = \frac{1}{K}\sum_{k=1}^{K} x^{(k)}_{n_k}$, is chosen as a global approximation of $\theta$ at time $n - 0$. Under the usual assumptions on function $r$ and error $e(n, x)$, not repeated here, and under the independence of delays $t(n)$, the normed approximation $n^{\frac{1}{2}}(\theta_n - \theta)$ is asymptotically normally distributed with parameters 0 and $\sigma/e$, where $\sigma^2$ is the asymptotic variance of the same normed approximation in a procedure without delays. Hence, $e$ is also the relative asymptotic efficiency of $\theta_n$ as a statistical estimator.

### *2.2. Description of the first loss system with delayed series of observations*

Consider the service system $GI/GI/K/0$ where both the inter-arrival and the service times are non-negative integer values, but their distributions are unspecified, otherwise. $K$ is the total number of servers (or of parallel service stations) in the system. 0 means that there are no waiting places and series are lost if all servers are busy. We will confine ourselves to the case of purely deterministic inter-arrival times, where one series of observations comes each time unit $n = 1, 2, \ldots$ sharp. Such a service system is called a loss system with delayed series of observations and is denoted by $D/GI/K/0$. The service time $t$ is assumed to be an integer valued random variable that however could have originated from a continuous one by off-rounding. The service time $t$ will be rounded down to 0, if the service of a series of observations, who came at time $n$, is finished by or immediately before the time $n + 1$ (i.e., at time $n + 1 - 0$), rounded down to 1, if the service is finished by the time $n + 2$ but not before time $n + 1$, etc.. Apparently, rounding up (to the next larger integer) would be more natural, but we are inspired by the application treated later on, where a service time not exceeding one time is considered as standard, and where $t$ plays the role of a delay, i.e. of an excess over one time unit.

Denote by $P_0; P_{1,}; P_2; \ldots$ the distribution of the rounded down service time of a series of observations or by $P_0; P_{1,}; P_2, \ldots; P_T$, if the (rounded down) service time, of a series of observations, cannot exceed $T$ time units.