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# Efficient quadrature rules for subdivision surfaces in isogeometric analysis

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**Abstract**

We introduce a new approach to numerical quadrature on geometries defined by subdivision surfaces based on quad meshes in the context of isogeometric analysis. Starting with a sparse control mesh, the subdivision process generates a sequence of finer and finer quad meshes that in the limit defines a smooth subdivision surface, which can be of any manifold topology. Traditional approaches to quadrature on such surfaces rely on per-quad integration, which is inefficient and typically also inaccurate near vertices where other than four quads meet. Instead, we explore the space of possible groupings of quads and identify the optimal macro-quads in terms of the number of quadrature points needed. We show that macro-quads consisting of quads from one or several consecutive levels of subdivision considerably reduce the cost of numerical integration. Our rules possess a tensor product structure and the underlying univariate rules are Gaussian, i.e., they require the minimum possible number of integration points in both univariate directions.

The optimal quad groupings differ depending on the particular application. For instance, computing surface areas, volumes, or solving the Laplace problem lead to different spline spaces with specific structures in terms of degree and continuity. We show that in most cases the optimal groupings are quad-strips consisting of $(1 \times n)$ quads, while in some cases a special macro-quad spanning more than one subdivision level offers the most economical integration.

Additionally, we extend existing results on exact integration of subdivision splines. This allows us to validate our approach by computing surface areas and volumes with known exact values. We demonstrate on several examples that our quadratures use fewer quadrature points than traditional quadratures. We illustrate our approach to subdivision spline quadrature on the well-known Catmull-Clark scheme based on bicubic splines, but our ideas apply also to subdivision schemes of arbitrary bidegree, including non-uniform and hierarchical variants. Specifically, we address the problems of computing areas and volumes of Catmull-Clark subdivision surfaces, as well as solving the Laplace and Poisson PDEs defined over planar unstructured quadrilateral meshes in the context of isogeometric analysis.

*Keywords:* Numerical integration, subdivision surface, non-tensor-product splines, Gaussian quadrature rules, isogeometric analysis.

## 1. Introduction

Subdivision surfaces [39] are a popular modelling tool due to their ability to represent shapes of arbitrary manifold topology and are the representation of choice in 3D animated films [17]. The most popular subdivision scheme is that developed by Catmull and Clark [12]. Subdivision surfaces have also been used in the context of numerical analysis. The seminal papers [13, 21] in this area are based on the subdivision scheme of Loop [30] and pre-date the advent of isogeometric analysis (IgA) [15].

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