Available online at www.sciencedirect.com
ScienceDirect
Comput. Methods Appl. Mech. Engrg. 333 (2018) 55–73
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

A quasi-Lagrangian finite element method for the Navier–Stokes equations in a time-dependent domain[☆]

Alexander Lozovskiy[a], Maxim A. Olshanskii[b,*], Yuri V. Vassilevski[c]

[a] Institute of Numerical Mathematics RAS, Russian Federation
[b] Department of Mathematics, University of Houston, USA
[c] Institute of Numerical Mathematics RAS, Moscow Institute of Physics and Technology, Sechenov University, Russian Federation

Received 17 July 2017; received in revised form 12 January 2018; accepted 17 January 2018

Abstract

The paper develops a finite element method for the Navier–Stokes equations of incompressible viscous fluid in a time-dependent domain. The method builds on a quasi-Lagrangian formulation of the problem. The paper provides stability and convergence analysis of the fully discrete (finite-difference in time and finite-element in space) method. The analysis does not assume any CFL time-step restriction, it rather needs mild conditions of the form $\Delta t \le C$, where C depends only on problem data, and $h^{2m_u+2} \le c\,\Delta t$, m_u is polynomial degree of velocity finite element space. Both conditions result from a numerical treatment of practically important non-homogeneous boundary conditions. The theoretically predicted convergence rate is confirmed by a set of numerical experiments. Further we apply the method to simulate a flow in a simplified model of the left ventricle of a human heart, where the ventricle wall dynamics is reconstructed from a sequence of contrast enhanced computed tomography images.

Keywords: Incompressible Navier–Stokes; Moving domain; Finite elements; Error analysis

1. Introduction

Fluid flows in time-dependent domains are ubiquitous in nature and engineering. In many cases, finding the domain evolution is part of the problem and the mathematical model couples fluid and structure dynamics. Examples include fluid–structure interaction problems for blood flow in compliant vessels, flows around turbine blades or fish locomotion. In other situations, one may assume that the motion of the domain is given and one has to recover the induced fluid flow. One example of a problem, which often assumes *a priori* information about the flow domains evolution, is the blood flow simulation in a human heart when the (patient-specific) motion of the heart walls is

☆ This work has been supported by the Russian Science Foundation (RSF) grant 14-31-00024.
* Corresponding author.
E-mail addresses: saiya-jin@yandex.ru (A. Lozovskiy), molshan@math.uh.edu (M.A. Olshanskii), yuri.vassilevski@gmail.com (Y.V. Vassilevski).

https://doi.org/10.1016/j.cma.2018.01.024
0045-7825/

recovered from a sequence of medical images [1–8]. Nowadays numerical simulations are commonly used to understand fluid dynamics and predict statistics of practical interest in this and other applications. In the present paper, we develop a finite element (FE) method for a quasi-Lagrangian formulation of the incompressible Navier–Stokes equations in a moving domain. We consider an implicit–explicit method, i.e. an implicit method with advection field in the inertia term lagged in time. For the spatial discretization we employ inf–sup stable pressure–velocity elements.

Several techniques have been introduced in the literature to overcome numerical difficulties due to the evolution of the domain. This includes space–time finite element formulations, immersed boundary methods, level-set method, fictitious domain method, unfitted finite elements, and arbitrary Lagrangian–Eulerian (ALE) formulation, see, e.g., [9–17]. In this paper we analyze a finite element method based on a quasi-Lagrangian formulation of the equations in the reference domain. Related analysis of finite element methods for parabolic or fluid equations in moving domains can be found in several places in the literature. We note that well-posedness of space–time weak saddle-point formulations of the (Navier–)Stokes equations is a subtle question, see the recent treatment in [18] for the case of a steady domain. A rigorous stability and convergence analysis of space–time (FE) methods for fluid problems seems to be largely lacking. Scalar problems have been understood much better; for example, a space–time discontinuous FE method for advection–diffusion problems on time-dependent domains was analyzed in [19]. ALE and Lagrangian finite element methods are more amenable to analysis. The stability of ALE finite element methods for parabolic evolution problems was treated in [16]. The authors of [20] analyzed the convergence of a finite element ALE method for the Stokes equations in a time-dependent domain when the motion of the domain is given. The analysis [20] imposes time step restriction, assumes zero velocity boundary condition and certain smoothness assumptions for the finite element displacement field. A closely related method to the one studied here was considered in [21]. However, that paper introduced an assumption that a divergence free extension of a boundary condition function to the computational domain is given. This assumption is not always practical and the present paper avoids it. Moreover, this paper develops error analysis, while the thrust of [21] was the stability analysis and the numerical recovery procedure of the domain motion from medical images.

In the present paper, we analyze a quasi-Lagrangian FE formulation that is closely related to an ALE formulation, although they are not equivalent. In the present approach we discretize equations in a reference time-independent domain. The geometry evolution is accounted in time-dependent coefficients. Inertia terms are further linearized so that only a system of linear algebraic equations is solved in each time step. We consider practically relevant boundary conditions, which result in non-homogeneous velocity on the boundary. For this method we prove numerical stability and optimal order error estimates in the energy norm without a CFL condition in the time step. Divergence-free condition enforced in the reference domain leads to time dependent functional spaces; this and handling non-homogeneous boundary conditions are two main difficulties that we overcome in the analysis. For the numerical stability bound we shall need the condition on the space mesh size and time step of the form $h^{2m_u+2} \le c\,\Delta t$, where $m_u \ge 1$ is polynomial degree of velocity finite element space and c is a constant. We note that if one assumes zero boundary conditions for velocity, which is a standard assumption for FE stability bounds in steady domains, then the results of the paper hold without the above conditions on h and Δt. In our opinion, homogeneous boundary conditions are not a suitable assumption for the FE analysis in evolving domains, see discussion in Section 3.

Thus the paper advances the known analysis by including inertia effect, removing CFL time-step restriction, handling physically meaningful boundary conditions, and making no further auxiliary assumptions except the following one: the domain evolution is given *a priori* by a smooth mapping from a reference domain to a physical domain and exact quadrature rules are applied in the reference domain, i.e. we do not analyze possible errors due to inexact numerical integration. The mapping is *not* necessarily Lagrangian in the internal points, but it has to be Lagrangian for those parts of the boundary, where correct tangential velocity boundary values are important. Theoretical results are illustrated numerically for an example with a synthetic known solution. We further illustrate the performance of the numerical method by applying it to blood flow simulation in a simplified model of the human left ventricle. The domain motion in this example is reconstructed from a sequence of CECT images of a real patient heart over one cardiac cycle. The reconstruction procedure is described in detail in our preceding paper [21].

The remainder of this paper is organized as follows. In Section 2 we review the mathematical model, including governing equations and boundary conditions, and some useful results for this model found in the literature. We recall the energy balance satisfied by smooth solutions. A suitable weak formulation is introduced. Based on the weak formulation, in Section 3 we introduce the finite element method. Non-homogeneous boundary conditions are interpolated numerically. Energy stability estimate for the finite element method is shown in Section 4. Optimal order error bound for the method is demonstrated in Section 5. Section 6 collects results of numerical experiments. Some closing remarks can be found in the summary and outlook Section 7.