Available online at www.sciencedirect.com
ScienceDirect
Comput. Methods Appl. Mech. Engrg. 305 (2016) 217–240
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

Optimal quadrature rules for odd-degree spline spaces and their application to tensor-product-based isogeometric analysis

Michael Bartoň[b,a,*], Victor Manuel Calo[c,a]

[a] Center for Numerical Porous Media, King Abdullah University of Science and Technology, Thuwal 23955-6900, Saudi Arabia
[b] Basque Center for Applied Mathematics, Alameda de Mazarredo 14, 48009 Bilbao, Basque Country, Spain
[c] Department of Applied Geology, Western Australian School of Mines, Faculty of Science and Engineering, Curtin University, Kent Street, Bentley, Perth, Western Australia, 6102, Australia

Received 12 November 2015; received in revised form 22 February 2016; accepted 23 February 2016
Available online 14 March 2016

Abstract

We introduce optimal quadrature rules for spline spaces that are frequently used in Galerkin discretizations to build mass and stiffness matrices. Using the homotopy continuation concept (Bartoň and Calo, 2016) that transforms optimal quadrature rules from source spaces to target spaces, we derive optimal rules for splines defined on finite domains. Starting with the classical Gaussian quadrature for polynomials, which is an optimal rule for a discontinuous odd-degree space, we derive rules for target spaces of higher continuity. We further show how the homotopy methodology handles cases where the source and target rules require different numbers of optimal quadrature points. We demonstrate it by deriving optimal rules for various odd-degree spline spaces, particularly with non-uniform knot sequences and non-uniform multiplicities. We also discuss convergence of our rules to their asymptotic counterparts, that is, the analogues of the midpoint rule of Hughes et al. (2010), that are exact and optimal for infinite domains. For spaces of low continuities, we numerically show that the derived rules quickly converge to their asymptotic counterparts as the weights and nodes of a few boundary elements differ from the asymptotic values.

Keywords: Optimal quadrature rules; Galerkin method; Gaussian quadrature; B-splines; Isogeometric analysis; Homotopy continuation for quadrature

1. Introduction and motivation

Numerical integration is a basic ingredient of Galerkin discretizations, which are at the core of finite elements and isogeometric analysis. We aim to make this fundamental building block of computation more economical by introducing new optimal quadrature rules for numerical integration.

We derive optimal rules that result in important computational savings and can be used in many applied fields. Due to the popularity and success of isogeometric analysis in many relevant engineering applications [1–25], we focus on

* Corresponding author at: Basque Center for Applied Mathematics, Alameda de Mazarredo 14, 48009 Bilbao, Basque Country, Spain.
E-mail addresses: Michael.Barton@centrum.cz (M. Bartoň), Victor.Calo@Curtin.edu.au (V.M. Calo).

http://dx.doi.org/10.1016/j.cma.2016.02.034

spline spaces, which are piece-wise polynomial functions with controlled global smoothness [26,27]. Thus, we derive optimal (Gaussian) quadrature rules for spline spaces of arbitrary order and continuity.

A spline space is uniquely determined by its *knot vector*. This knot vector is a non-decreasing sequence of real numbers called *knots* and the multiplicity of each knot determines the smoothness of the polynomial pieces at the knot location. A detailed introduction on splines can be found, e.g., in [28–30].

We call a *quadrature rule*, or simply a *quadrature*, an *m-point rule*, if m evaluations of a function f are needed to approximate its weighted integral over a closed interval $[a, b]$

$$\int_a^b w(x) f(x)\,\mathrm{d}x = \sum_{i=1}^{m} \omega_i f(\tau_i) + R_m(f), \tag{1}$$

where w is a fixed non-negative *weight function* defined over $[a, b]$. The rule is required to be *exact*, i.e., $R_m(f) \equiv 0$ for each element of a spline space S. A rule is *optimal* if m is the minimum number of *weights* ω_i and *nodes* τ_i (points at which f is evaluated).

For discontinuous spline spaces (polynomials), the optimal rule is known to be the classical Gaussian quadrature [31] with the *order of exactness* $2m-1$, that is, only m evaluations are needed to exactly integrate any polynomial of degree at most $2m-1$. Consider a sequence of polynomials $(q_0, q_1, \ldots, q_m, \ldots)$ that form an orthogonal basis with respect to the scalar product

$$\langle f, g\rangle = \int_a^b f(x)g(x)w(x)\mathrm{d}x. \tag{2}$$

The quadrature points are the roots of the mth orthogonal polynomial q_m. In the case when $w(x) \equiv 1$, this is the degree-m Legendre polynomial [32].

The quadrature rules for splines have been studied since the late 50s [33–35]. Micchelli and Pinkus [35] proved that, for spaces with uniform continuity, i.e. knots' multiplicity, there always exists an optimal quadrature formula with the following number of necessary evaluations:

$$d + 1 + i = 2m, \tag{3}$$

where d is the polynomial degree, i is the total number of interior knots (when counting multiplicities), and m is the number of optimal nodes.

Regarding concrete optimal quadrature rules over a finite domain, very little is known in the literature. Even though [35] gives a range of knot spans, among which each particular node lies, the set of subintervals is too large to even build the corresponding polynomial system. Therefore, formulating it as an optimization problem is rather difficult as it is highly non-linear and a good initial guess in essential. To the best of our knowledge, however, there is no theory that tightly bounds each node to give a good initialization for the numerical optimization.

For C^1 cubic splines with uniform knots, Nikolov [36] derived a recursive formula that starts at the boundary of the interval and parses towards its middle, recursively computing all the optimal nodes and weights. Computationally, this result is favorable as it computes the nodes and weights in a *closed form*, i.e., without the need of any numerical solver [37–39]. We have generalized this result to a special *class of spaces* defined on non-uniform knot vectors [40], called symmetrically stretched B-splines. Recently, we have described a similar recursive pattern as Nikolov for C^1 quintic spline spaces with uniform knots [41] and derived optimal rules for them. Interestingly, the optimal rules are still explicit, even though degree five polynomials are involved. We proved that there exists an algebraic factorization in every step of the recursion which makes the rule explicit and therefore computationally cheap.

Recently, we demonstrated a relation between Gaussian quadrature rules for C^1 and C^2 cubic splines [42]. We showed that for particular pairs of spaces that require the same number of optimal nodes, the quadrature rule can be "transferred" from the *source* space to the *target* space, preserving the number of internal knots in (3) and therefore the number of optimal quadrature points. This transfer relies on the application of homotopy continuation to preserve the algebraic structure of the system as the spline space evolves [43]. Starting with a known optimal rule for a C^1 cubic spline space, the source knot vector $\widetilde{\mathcal{X}}$ (with double knots) gets transformed to the desired configuration $\mathcal{X}$ (single knots). The quadrature rule is considered as a high-dimensional point, and satisfies a certain well-constrained system of polynomial equations. These equations reflect the fact that the rule must exactly integrate all the basis functions