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A computational framework for polyconvex large strain elasticity

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Highlights

- Conjugate stresses to the extended kinematic set, elegantly related to classical stress tensors.
- A convex complementary strain energy functional in terms of a new set of conjugate stresses.
- Formulation of several variational principles, expanding the ideas in Schröder et al. (2011).
- A novel Hellinger–Reissner variational principle.
- A stabilised Petrov–Galerkin discretisation in the context of mixed principles.

Abstract

This paper presents a novel computational formulation for large strain polyconvex elasticity. The formulation, based on the original ideas introduced by Schröder et al. (2011), introduces the deformation gradient (the fibre map), its adjoint (the area map) and its determinant (the volume map) as independent kinematic variables of a convex strain energy function. Compatibility relationships between these variables and the deformed geometry are enforced by means of a multi-field variational principle with additional constraints. This process allows the use of different approximation spaces for each variable. The paper extends the ideas presented in Schröder et al. (2011) by introducing conjugate stresses to these kinematic variables which can be used to define a generalised convex complementary energy function and a corresponding complementary energy principle of the Hellinger–Reissner type, where the new conjugate stresses are primary variables together with the deformed geometry. Both compressible and incompressible or nearly incompressible elastic models are considered. A key element to the developments presented in the paper is the new use of a tensor cross product, presented for the first time by de Boer (1982), page 76, which facilitates the algebra associated with the adjoint of the deformation gradient. For the numerical examples, quadratic interpolation of the displacements, piecewise linear interpolation of strain and stress fields and piecewise constant interpolation of the Jacobian and its stress conjugate are considered for compressible cases. In the case of incompressible materials two formulations are presented. First, continuous quadratic interpolation for the displacement together with piecewise constant interpolation for the pressure and second, linear continuous interpolation for both displacement and pressure stabilised via a Petrov–Galerkin technique.

Keywords: Large strain elasticity; Polyconvex elasticity; Mixed variational principle; Complementary energy variational principle; Incompressible elasticity; Finite elements

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1. Introduction

Large strain elastic and inelastic analysis by finite elements or other computational techniques is now well established for many engineering applications [1–8]. Often elasticity is described by means of a hyperelastic model defined in terms a stored energy functional which depends on the deformation gradient of the mapping between initial and final configurations [1,9–17]. It has also been shown that for the model to be well defined in a mathematical sense, this dependency with respect to the deformation gradient has to satisfy certain convexity criteria [1,12,13]. The most well-established of these criteria is the concept of polyconvexity [14–20] whereby the strain energy function must be expressed as a convex function of the components of the deformation gradient, its determinant and the components of its adjoint or cofactor. Numerous authors have previously incorporated this concept into computational models for both isotropic and non-isotropic materials for a variety of applications [21–26].

The classical approach consists of ensuring that the stored energy function satisfies the polyconvexity condition first but then proceed towards a computational solution by re-expressing the energy function in terms of the deformation gradient alone. More recently, Schröder et al. [24] have proposed a new mixed formulation in which the adjoint of the deformation gradient and its determinant are retained as fundamental problem variables by means of a Hu–Washizu type of mixed variational principle [27]. This pioneering formulation [24] opens up new interesting possibilities in terms of using various interpolation spaces for different variables [28–31], leading to enhanced type of formulations [24]. There is extensive literature in the field of mixed and enhanced formulations for large strain solid mechanics by numerous authors [32–43].

The present contribution aims to present a systematic framework for developing computational approaches for hyperelastic (or hyperelastic–plastic) materials governed by a polyconvex energy functional. Similarly to Refs. [24,44,45], the framework proposed is based on maintaining as independent variables the fundamental variables on which the strain energy is expressed as a convex function, namely, the deformation gradient, its adjoint and its determinant. These variables are, of course, not independent from each other but their relationships can be enforced by appropriate constraints in a multi-field variational principle leading to an extended Hu–Washizu type of variational principle [1,34].

In contrast with previous work, the paper proposes a novel algebra to deal with the additional kinematic variables, which is based on the new use of a tensor cross product operation, originally introduced by de Boer [46], page 76. To the best of the authors' knowledge, this is the first time that this tensor cross product is used within the context of continuum mechanics. This new notation, not only greatly simplifies the formulation but it also provides new expressions for operators such as the tangent elasticity tensor which are useful from a computational and theoretical point of view.

The paper also introduces a set of stress variables work conjugate to the extended set of independent kinematic variables [47]. As a result of the convexity with respect to the extended kinematic set, it is possible to define a complementary strain energy function which is convex with respect to the set of conjugate stresses. This new complementary strain energy makes it possible to introduce Hellinger–Reissner [34,39] type of mixed variational principles in the context of large strain elasticity, with a significantly reduced set of variables over a more traditional Hu–Washizu type of functional.

In order to ease the understanding of the new concepts proposed, this paper only considers a Mooney–Rivlin type constitutive model, both in the compressible and incompressible regime. This is the simplest model available which contains all the features of general polyconvex elasticity strain energy functions, that is a dependency with respect to the deformation gradient, its adjoint and its determinant in a convex manner. Extending the proposed formulation to more complex models is a simple algebraic exercise.

Note that it is not the primary aim of this paper to propose specific choices of interpolation spaces. Nevertheless, two examples of the use of the framework are provided. First, an application of the complementary energy functional with quadratic displacements and linear stresses is presented. This is followed by a model for incompressible elasticity using stabilised linear tetrahedral elements.

The paper is organised as follows. Section 2 introduces the novel tensor cross product notation in the context of large strain deformation. This definition is used to re-express the adjoint of the deformation gradient and its directional derivatives in a novel, simple and convenient manner. Section 3 reviews the definition of polyconvex elastic strain energy functions and defines a new set of stresses conjugate to the main kinematic variables. The relationships between these stresses and standard stress tensors such as the Piola–Kirchhoff stresses and Cauchy