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# A local solution approach for adaptive hierarchical refinement in isogeometric analysis

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**Highlights**

- The hierarchical refinement method is extended to handle parametric domains with repeated knot values.
- Coefficient matrices are utilized to describe the relations between basis functions at different levels.
- A new local solution method is proposed to solve hierarchical refinement problems.

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**Abstract**

Adaptively refining the state variables and fast solving the equations are two key issues in isogeometric analysis. Currently existing research more focuses on the first issue while the efficiency issue is not elaborated when the overall equations are solved at each refinement step. This article develops an approach to promote the efficiency by locally solving the equations that govern the refined variables at each refinement step. Here, hierarchical refinement of the state variables in NURBS is adopted and transformational matrix that transforms the basis functions from the current level to the refined level is utilized to handle the relations between the equations at different levels. By dividing the coefficient matrix into blocks and estimating the error of state variables for different blocks, we represent the global solution method in a hierarchical structure and separately construct and solve equations to obtain the solution for each level in the structure. We validate the proposed method by some steady heat-transfer and structural analysis problems in domains with different dimensionality. We have compared the numerical result accuracy as well as the time cost of the proposed method with those of current available refinement approaches, and it turns out to be very promising.

*Keywords:* Isogeometric analysis; NURBS; Adaptive refinement; Solution refinement

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## 1. Introduction

Isogeometric analysis (IGA) as a new CAE paradigm that uses the same basis functions for representations of geometric shapes and physical fields was introduced by Hughes et al. [1] and it has a potential capability to make up the gap between CAD and finite element analysis (FEA). Compared with the classic FEA, IGA has three aspects

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of superiority. First, the geometry in the classic FEA is approximated by piecewise linear or higher-order elements, which fail to exactly represent the geometry and usually might require undesirably huge number of elements for a good approximation of the geometry. In contrast, IGA directly uses the geometric data from the geometry's representations as well as the mathematical expressions constructed with the data to form the expressions of analysis solutions, which avoids the costly and inaccurate data transformations from design to analysis. Second, based on the spline representations for analysis, it is possible to achieve more precise numerical solutions with coarser meshes and less control points in IGA. Third, the classic FEA requires the repeated communications with the CAD programs when its mesh needs to be updated for a better geometric approximation. However, in IGA, due to directly using the geometric information, the further communication with CAD programs is not necessary, which may reduce the time required for the process of meshing, especially for the refinement process. For the analysis of complex engineering design, this could potentially reduce the time by up to 80% [2].

Because of these differences, IGA has its different refinement methods. There exist two main refinement methods for FEA, $p$-refinement and $h$-refinement. However, the refinement of IGA, which relies on the principle of knot insert of parameters, does not need to refine its physical model directly but the space of parametric knot vectors. The sizes of elements or the number of elements depends on distances of two adjacent unequal knot nodes in knot vectors. There are three refinement methods for IGA: the knot insertion corresponding to $h$-refinement in FEA, the order elevation corresponding to $p$-refinement in FEA, and $k$-refinement [1] which is a spline-specific method. Cottrell et al. [3] compared $h$-, $p$- and $k$-refinement and showed that $k$-refinement produced a better accuracy than the other methods on degree-of-freedom basis. These refinement methods can be further classified into the global and local refinement methods according to the range of the refined domain, or adaptive and non-adaptive refinement methods based on the way to determine the refinement levels.

Due to the tensor-product structure of B-splines, it is difficult to express a complex geometric shape with one B-spline surface. To overcome this problem, the hierarchical spline was proposed. The concept of hierarchical B-splines was introduced by Forsey and Bartels [4,5] to construct the geometric models for grotesque figures with rich local features in computer graphics and later adopted by Höllig and his co-workers [6,7] for the local refinement in analysis when B-splines are used to express the physical field solution. The hierarchical splines offer a local refinement means through the use of overlays and the application range is extended to large curvature problems. Studies on the IGA that is based on hierarchical models have been applied to a wide range of areas, such as elasticity [8], shells, fluid–structure interaction [9], acoustics [10], electromagnetic, fracture and damage [11,12], and even the contact problems [13]. Recently, hierarchical refinement and adaptive hierarchical refinement of NURBS and T-spline, which gives a watertight geometric representation for compound objects and uses fewer control points for common complex shapes [14–16], have become an active research topic within the framework of IGA.

Hierarchical refinement has gained huge attention during the past few years [1]. Schillinger et al. [17] introduce a B-spline finite element method, in which the discontinuities are adaptively approximated with hierarchical grid refinement and the interface issue in complex geometry is focused as well. In the close vicinity of interfaces, several levels of local basis functions are added to maintain the regular grid of knot span elements. In [18], Schillinger et al. explored the hierarchical refinement of NURBS as a basis for adaptive isogeometric and immersed boundary analysis, and used the principle of B-spline subdivision to derive a local refinement procedure. Through two examples respectively for ship propeller and automobile wheel, the authors illustrate an isogeometric design-through-analysis procedure for complex engineering problems in detail. Bornemann and Cirak [19] proposed a more intuitional method for hierarchical refinement where the algebraic projection relations of the basis functions on different levels are established and the boundary associations between refining and refined domains are dealt with using the relations. Schillinger et al. [20] have compared the computational efficiency of isogeometric collocation (IGA-C) with that of isogeometric Galerkin (IGA-G) and standard $C^0$ finite element methods (FEA-G). With the comparison metrics that take into account the cost of direct and iterative solvers, the accuracy versus degrees of freedom and the accuracy versus computing time, the authors reached a conclusion that IGA-C is several orders of magnitude faster than the two other methods to achieve a specified level of accuracy. Scott et al. [21] and Thomas et al. [22] recently proposed a concept of spline forest and a unified approach for refinement and coarsening of splines based on Bézier projection. They developed a Bézier extraction framework which simplifies the finite element description in the complex multi-level, unstructured hierarchical splines.

Besides the refinement issue, the solution efficiency is another major concern in the implementation of IGA to practical engineering problems, though it does not receive much attention from the scholars in the refinement research area.