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Further insights into an implicit time integration scheme for structural dynamics

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ABSTRACT

The objective in this paper is to present some new insights into an implicit direct time integration scheme, the Bathe method, for the solution of the finite element equations of structural dynamics. The insights pertain to the use of the parameters in the method, and in particular the value of the time step splitting ratio. We show that with appropriate values of this ratio large amplitude decays can be obtained as may be desirable in some solutions. We give the theoretical analysis of the method for the parameters used, including for very large time steps, and illustrate numerically the new insights gained.

1. Introduction

The finite element method is now quite widely used for the solution of structural vibrations and wave propagations. For such solutions, methods of direct time integration of the governing finite element equations are frequently used. When so done, for wave propagation analyses, mostly explicit schemes are employed, while for structural vibration solutions mostly implicit schemes are used [1]. While such use is still mostly the case, lately implicit schemes are also increasingly being employed to solve wave propagation problems.

An important characteristic of an implicit direct time integration scheme is that it can be unconditionally stable, which means that the time step Δt can be selected solely based on accuracy considerations. For the solution of the finite element structural dynamics equations, a frequently used scheme is the trapezoidal rule, a special case of the Newmark method [1,2]. Some other techniques are, for example, the Wilson method [1,3,4], the methods discussed by Zhou and Tamma [5], and the three parameter method [6–8]. More recently, the Bathe method has been proposed and has attracted significant attention and use [9–11].

The Bathe method is a composite scheme using two sub-steps per time step Δt, but *the method is used like a single-step method with some intermediate calculations*. These, however, make the method about twice as expensive as the trapezoidal rule per time step Δt. Since there seems to be a considerable extra cost compared to using other methods, an analyst may question why to use this method. The advantages are much better accuracy characteristics allowing larger time steps in the integration and, overall, the more effective solution of linear and nonlinear problems. The method is now widely used for structural analyses and fluid-structure interactions, see e.g., Refs. [12,13].

The objective in this paper is to focus on some considerations regarding the method, observations and insights that are related to what we have learned since the first publication of the method. The new insights are based on our latest experiences and thoughts spurred by other publications [13–21]. Hence, this paper might be viewed as a continuation of Ref. [11].

In the next section, we briefly summarize the basic equations of the Bathe method. As originally published, there are three parameters (α, δ, γ) that can be set. Although it is in practical analyses frequently best to simply use the defaults of these parameters, as emphasized in Refs. [10,11], it is important to realize that, if desired, the three parameters can of course be changed by the analyst. Of particular interest are the properties when γ is changed. We therefore discuss, in Sections 3 and 4, the stability and accuracy of the method when changing γ, where we include the use of γ larger than 1.0 and large time step to period ratios. Finally, in Section 5, we present our concluding remarks.

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2. The basic equations of the Bathe time integration scheme

To provide a basis for our discussion, we summarize here briefly the governing equations of the Bathe method [9–11]. In the procedure, we calculate the unknown displacements, velocities, and accelerations by considering the time step Δt to consist of two sub-steps. The sub-step sizes are $\gamma\Delta t$ and $(1-\gamma)\Delta t$ for the first and second sub-steps, respectively. Of course, in principle more sub-steps could be employed, see for example Ref. [9].

In the Bathe method, well-known integration schemes are used for each sub-step [10]. In the first sub-step, we utilize the trapezoidal rule, or more generally, the Newmark method

$$ {}^{t+\gamma\Delta t}\dot{\mathbf{U}} = {}^{t}\dot{\mathbf{U}} + [(1-\delta){}^{t}\ddot{\mathbf{U}} + \delta\, {}^{t+\gamma\Delta t}\ddot{\mathbf{U}}]\gamma\Delta t \tag{1} $$

$$ {}^{t+\gamma\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + {}^{t}\dot{\mathbf{U}}\gamma\Delta t + \left[\left(\frac{1}{2}-\alpha\right){}^{t}\ddot{\mathbf{U}} + \alpha\, {}^{t+\gamma\Delta t}\ddot{\mathbf{U}}\right]\gamma^2\Delta t^2 \tag{2} $$

and in the second sub-step, we utilize the Euler 3-point backward rule

$$ {}^{t+\Delta t}\dot{\mathbf{U}} = c_1{}^{t}\mathbf{U} + c_2{}^{t+\gamma\Delta t}\mathbf{U} + c_3{}^{t+\Delta t}\mathbf{U} \tag{3} $$

$$ {}^{t+\Delta t}\ddot{\mathbf{U}} = c_1{}^{t}\dot{\mathbf{U}} + c_2{}^{t+\gamma\Delta t}\dot{\mathbf{U}} + c_3{}^{t+\Delta t}\dot{\mathbf{U}} \tag{4} $$

where

$$ c_1 = \frac{1-\gamma}{\gamma\Delta t}, \quad c_2 = \frac{-1}{(1-\gamma)\gamma\Delta t}, \quad c_3 = \frac{2-\gamma}{(1-\gamma)\Delta t} \tag{5} $$

Considering linear analysis, the equilibrium equations applied at time $t+\gamma\Delta t$ and time $t+\Delta t$ are

$$ \mathbf{M}\,{}^{t+\gamma\Delta t}\ddot{\mathbf{U}} + \mathbf{C}\,{}^{t+\gamma\Delta t}\dot{\mathbf{U}} + \mathbf{K}\,{}^{t+\gamma\Delta t}\mathbf{U} = {}^{t+\gamma\Delta t}\mathbf{R} \tag{6} $$

$$ \mathbf{M}\,{}^{t+\Delta t}\ddot{\mathbf{U}} + \mathbf{C}\,{}^{t+\Delta t}\dot{\mathbf{U}} + \mathbf{K}\,{}^{t+\Delta t}\mathbf{U} = {}^{t+\Delta t}\mathbf{R} \tag{7} $$

where $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ are the mass, damping and stiffness matrices, and the vectors $\mathbf{U}$ and $\mathbf{R}$ list, respectively, the nodal displacements (rotations) and externally applied nodal forces (moments). An overdot denotes a time derivative. Using the relations in Eqs. (1)–(5) with the equilibrium equations at the two time points, Eqs. (6) and (7), we construct the time-stepping equations as

$$ \hat{\mathbf{K}}_1{}^{t+\gamma\Delta t}\mathbf{U} = \hat{\mathbf{R}}_1 \tag{8} $$

$$ \hat{\mathbf{K}}_2{}^{t+\Delta t}\mathbf{U} = \hat{\mathbf{R}}_2 \tag{9} $$

where

$$ \hat{\mathbf{K}}_1 = \frac{1}{\alpha\gamma^2\Delta t^2}\mathbf{M} + \frac{\delta}{\alpha\gamma\Delta t}\mathbf{C} + \mathbf{K} \tag{10} $$

$$ \hat{\mathbf{K}}_2 = {c_3}^2\mathbf{M} + c_3\mathbf{C} + \mathbf{K} \tag{11} $$

$$ \begin{aligned} \hat{\mathbf{R}}_1 = {}&{}^{t+\gamma\Delta t}\mathbf{R} + \mathbf{M}\left(\left(\frac{1}{2\alpha}-1\right){}^{t}\ddot{\mathbf{U}} + \frac{1}{\alpha\gamma\Delta t}{}^{t}\dot{\mathbf{U}} + \frac{1}{\alpha\gamma^2\Delta t^2}{}^{t}\mathbf{U}\right) \\ &+ \mathbf{C}\left(\frac{(\delta-2\alpha)\gamma\Delta t}{2\alpha}{}^{t}\ddot{\mathbf{U}} + \left(\frac{\delta}{\alpha}-1\right){}^{t}\dot{\mathbf{U}} + \frac{\delta}{\alpha\gamma\Delta t}{}^{t}\mathbf{U}\right) \end{aligned} \tag{12} $$

$$ \begin{aligned} \hat{\mathbf{R}}_2 = {}&{}^{t+\Delta t}\mathbf{R} - \mathbf{M}(c_1c_3{}^{t}\mathbf{U} + c_2c_3{}^{t+\gamma\Delta t}\mathbf{U} + c_1{}^{t}\dot{\mathbf{U}} + c_2{}^{t+\gamma\Delta t}\dot{\mathbf{U}}) \\ &- \mathbf{C}(c_1{}^{t}\mathbf{U} + c_2{}^{t+\gamma\Delta t}\mathbf{U}) \end{aligned} \tag{13} $$

While the case of the trapezoidal rule ($\alpha = 1/4, \delta = 1/2$) as the first sub-step with $\gamma = 1/2$ (or $\gamma = 2-\sqrt{2}$) has been mostly used and analyzed, in principle, α and δ in the Newmark method and γ in the composite scheme can be chosen. Hence we can interpret the scheme as a "quite general time stepping technique."

To have a second-order accurate composite procedure with the second-order accurate Euler 3-point backward rule for the second sub-step, it is apparent that the scheme in the first sub-step should have at least second-order accuracy. Hence in the Newmark method used in the Bathe scheme, δ must be equal to 1/2, which gives second-order accuracy of the displacements, velocities and accelerations and also no dissipation. This has been proven explicitly in Ref. [20]. However, not seeking second-order accuracy and choosing other values of δ, we may well have that with some values of δ, α and γ good solution accuracy for some problems is achieved. This approach may be valuable to obtain problem-fitted accuracy characteristics (e.g. larger numerical dissipation).

Considering nonlinear analysis, for some specific nonlinear problems, the parameters $\gamma = 0.73$, $\alpha = 1/4$ and $\delta = 1/2$ were identified to be suitable [17]. Also, for two nonlinear problems, the effect of choosing different values of γ with $\alpha = 1/4$ and $\delta = 1/2$ has been briefly studied in Ref. [16], and it was found that the use of different sets of parameters for different classes of nonlinear problems will likely yield a near-optimal time stepping measured on computational cost. However, to identify these sets of optimal parameters requires considerable analysis and experimentation. On the other hand, the insights that we can gain in linear analysis are also very useful for obtaining effective solutions in nonlinear analysis.

Note that the splitting ratio, γ, has been set to $\gamma \in (0,1)$ in the analysis and use of the Bathe method, as referred to above, and in the development of a new scheme based on the Bathe composite strategy [14,18,19,21]. However, in principle, the value of γ can also be defined in a larger range. In the following sections, we therefore also study the stability and accuracy of the method allowing γ to be greater than 1. Using γ greater than 1 is quite different from using θ greater than 1 in the Wilson method, because in the Bathe method the equilibrium equations are satisfied at both time points considered, see Eqs. (6) and (7).

3. Stability and accuracy of the Bathe method

In the decoupled modal equations, the method may be expressed as [1]

$$ \begin{bmatrix} {}^{t+\Delta t}\ddot{x} \\ {}^{t+\Delta t}\dot{x} \\ {}^{t+\Delta t}x \end{bmatrix} = \mathbf{A}\begin{bmatrix} {}^{t}\ddot{x} \\ {}^{t}\dot{x} \\ {}^{t}x \end{bmatrix} + \mathbf{L_a}{}^{t+\gamma\Delta t}r + \mathbf{L_b}{}^{t+\Delta t}r \tag{14} $$

where $\mathbf{A}$, $\mathbf{L_a}$ and $\mathbf{L_b}$ are the integration approximation and load operators, respectively, and x and an overdot denote the displacement in the modal space and a time derivative, respectively (see Appendix A). The stability and some accuracy characteristics of the method may be studied using this form of the scheme.

Focusing on the case of no physical damping, the characteristic polynomial of $\mathbf{A}$ is

$$ p(\lambda) = \lambda^3 - 2A_1\lambda^2 + A_2\lambda - A_3 \tag{15} $$

where

$$ A_1 = \frac{1}{\beta_{01}\beta_{02}}\left(\begin{array}{l} (1/4)\gamma^2(2\alpha-\delta)(\gamma-1)\Omega_0^4 \\ +\left(\begin{array}{l} -1+\gamma^4\alpha-4\gamma^3\alpha+(1/4)(16\alpha+2\delta+1)\gamma^2 \\ +(1/4)(-4\delta+2)\gamma \end{array}\right)\Omega_0^2+(\gamma-2)^2 \end{array}\right); $$

$$ A_2 = \frac{1}{\beta_{01}\beta_{02}}\left(\left(\begin{array}{l} \gamma^4\alpha-4\gamma^3\alpha+(1/2)(8\alpha+2\delta+1)\gamma^2 \\ +(1/2)(-4\delta-2)\gamma+1 \end{array}\right)\Omega_0^2+(\gamma-2)^2\right); \tag{16} $$

$$ A_3 = 0; $$

$$ \beta_{01} = \Omega_0^2\alpha\gamma^2 + 1; \quad \beta_{02} = (\gamma-1)^2\Omega_0^2 + (\gamma-2)^2 $$