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Shear deformable plate elements based on exact elasticity solution

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ABSTRACT

The 2-D approximation functions based on a general exact 3-D plate solution are used to derive locking-free, rectangular, 4-node Mindlin (i.e., first-order plate theory), Levinson (i.e., a third-order plate theory), and Full Interior plate finite elements. The general plate solution is defined by a biharmonic mid-surface function, which is chosen for the thick plate elements to be the same polynomial as used in the formulation of the well-known nonconforming thin Kirchhoff plate element. The displacement approximation that stems from the biharmonic polynomial satisfies the static equilibrium equations of the 2-D plate theories at hand, the 3-D Navier equations of elasticity, and the Kirchhoff constraints. Weak form Galerkin method is used for the development of the finite element model, and the matrices for linear bending, buckling and dynamic analyses are obtained through analytical integration. In linear buckling problems, the 2-D Full Interior and Levinson plates perform particularly well when compared to 3-D elasticity solutions. Natural frequencies obtained suggest that the optimal value of the shear correction factor of the Mindlin plate theory depends primarily on the boundary conditions imposed on the transverse deflection of the 3-D plate used to calibrate the shear correction factor.

1. Introduction

Shear deformation plate theories provide dimensionally reduced models for the structural analysis of flat solid bodies that may be moderately thick. The simplest among these theories is usually attributed to Mindlin and assumes that the transverse shear stresses are constant throughout the plate thickness [1–4]. The discrepancy between the predicted and actual shear behavior is corrected with an extrinsic shear correction factor. The Mindlin plate theory has survived decades of engineering practice showing that it is easy to use and gives accurate results for a wide range of real-life problems. The paper by Hrabok and Hrudey [5] delivers an overview on the early finite element developments that first extended the application of the Mindlin and classical Kirchhoff plate theories to modern, geometrically complex engineering structures. The Mindlin plate or, more aptly, shell finite elements that are nowadays used by swarms of engineers through software like Abaqus, Ansys and LS-DYNA are largely based on the works of Hughes et al. [6–10] and Belytschko et al. [11–14]. Another branch of shell elements that can be found, for example, in Adina, are the MITC elements by Bathe et al. [15–18].

Regardless of the sweeping success of the Mindlin plate theory, which is also known as the first-order shear deformation plate theory (FSDT) due to a linear displacement variation through the plate thickness, it is sometimes necessary to use a plate theory that describes more accurately the actual plate displacements, strains and stresses. To this end, the third-order shear deformation plate theory (TSDT) by Reddy [19,20] offers an alternative to the FSDT, for example, when the interlaminar stresses of a composite plate are of interest, by accommodating quadratic variations of the transverse shear stresses with respect to the plate thickness coordinate. Moreover, the TSDT does not require a shear correction factor, the determination of which can be cumbersome for composite plates, in particular. However, the total differential order of the governing equations of the TSDT is higher than that of the FSDT causing the analysis of the TSDT to be more laborious. In terms of finite elements, a typical four-node plate element based on the FSDT has three degrees of freedom at each node, whereas an element founded on the TSDT has five degrees of freedom (four rotations) at each node requiring ultimately considerably more computational effort.

We find that it would be convenient to have a plate model that both carries the benefits of Reddy's TSDT, and retains the simple mathematical structure of Mindlin's FSDT. It is worth noting that if the latter feature is achieved, many of the analytical and numerical methods applicable in the context of the Mindlin plate theory,

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including recent isogeometric developments [21–26], could be applied with little additional effort to the new plate model. Furthermore, the inclusion of the new model into commercial finite element software could be carried out following the footsteps of current implementations. We recently showed that it is possible to develop, within a well-defined interior framework, such plate models that combine the pros of the FSDT and TSDT [27]. We found that the Mindlin plate theory is in fact a special case of a general interior elasticity solution for a linearly elastic three-dimensional plate. Moreover, the solution includes third-order models, namely, Levinson and Full Interior plates, as other special cases. The equilibrium equations in terms of stress resultants are of the same form for all three theories and they do not include any higher-order stress resultants kin to those in the TSDT. In the present paper, we develop rectangular finite elements for the shear deformable plate models that are included in the 3-D interior elasticity solution.

In the remaining sections, we first present a brief overview on the interior framework for plates on the basis of our earlier works on the topic [27,28]. This is followed by the formulation of the rectangular finite elements for Mindlin, Levinson, and Full Interior plates by Galerkin's method of weighted residuals. The shape functions are obtained from the 3-D elasticity solution and they are the same for each 2-D element. The stiffness and consistent mass and geometric stiffness matrices are attained in closed-form through analytical integration. The shape functions satisfy the Kirchhoff constraints exactly so that each derived element reduces to a Kirchhoff element when the plate thickness tends to zero. Finally, buckling and natural frequency eigenproblems are studied using the novel, locking-free rectangular plate elements, and the performance and accuracy of the 2-D Mindlin, Levinson and Full Interior elements against each other and against 3-D plate solutions are evaluated. For further understanding on shear deformable plate theories, the physical crux of the numerical studies is to take a detailed look at the meaning of the shear correction factor of the Mindlin plate. The value of this factor is shown to depend notably on the boundary conditions of the 3-D plate which is used to calibrate the factor.

2. Overview on interior plates

Here, we first consider a plate with stress-free top and bottom faces and then focus on the interior stress state of the plate. The general 3-D solution to the interior problem is reviewed and presented in the conventional form of 2-D plate theories. We discuss the total potential energy of plates without boundary layers.

2.1. Starting point – stress-free faces and the interior solution

Let us consider a three-dimensional linearly elastic, isotropic, homogeneous plate of constant thickness h in Cartesian xyz-coordinate system. The stress boundary conditions on the faces of the plate read

$$\sigma_z(x,y,\pm h/2) = \tau_{xz}(x,y,\pm h/2) = \tau_{yz}(x,y,\pm h/2) = 0. \tag{1}$$

The most general state of stress within this plate can be decomposed into three parts: (1) interior state, (2) shear state, (3) Papkovich–Fadle state [29–31]. Detailed, general 3-D elasticity solutions for plates with stress-free faces which account for all these three states have been given by several authors [32–37]. It has been proven that both the shear and Papkovich–Fadle states are predominantly related to edge effects [29]. Our focus will be on the interior bending state, also known as the "plate theory part" [29].

The rectangular, linearly elastic interior plate of interest to us is depicted in Fig. 1. The length and width of the plate are $2a$ and $2b$, respectively. The general interior bending solution in terms of displacements can be written as [36]

$$2G\cdot U_x(x,y,z) = -z\frac{\partial\Psi}{\partial x} - \frac{1}{4(1-\nu)}\left[h^2 z - 2(2-\nu)\frac{z^3}{3}\right]\nabla^2\frac{\partial\Psi}{\partial x}, \tag{2}$$

$$2G\cdot U_y(x,y,z) = -z\frac{\partial\Psi}{\partial y} - \frac{1}{4(1-\nu)}\left[h^2 z - 2(2-\nu)\frac{z^3}{3}\right]\nabla^2\frac{\partial\Psi}{\partial y}, \tag{3}$$

$$2G\cdot U_z(x,y,z) = \Psi + \frac{\nu z^2}{2(1-\nu)}\nabla^2\Psi, \tag{4}$$

where G and ν are the shear modulus and Poisson ratio, respectively. In addition, we have

$$\nabla^4\Psi(x,y) = 0 \quad \left(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right). \tag{5}$$

Every term in the biharmonic mid-surface function $\Psi(x,y)$ (taken as a polynomial later) includes an arbitrary constant coefficient and these coefficients will correspond to the nodal degrees of freedom of the plate finite elements. The 3-D displacements U_x, U_y and U_z calculated from Eqs. (2)–(4) satisfy the 3-D Navier equations of elasticity.

At this point, the meaning of the "interior state" can be explained as follows. When all three parts of the general stress state are accounted for, boundary conditions at the outer edge of the boundary layer give rise to exponentially decaying edge effects. Once these edge effects have decayed entirely with distance from the edge, the interior solution prevails. In other words, the interior solution represents a plate section which has been cut out from a complete plate far enough from the actual lateral edge. The rationale for this description is well-embedded into the above solution – the third-order throughout-thickness displacement distributions of the interior plate are not suitable for the modeling of detailed boundary layer effects.

2.2. General 3-D solution in the form of 2-D plate theories

In order to present the 3-D solution (2)–(4) in the form of a 2-D plate theory, we define the transverse deflection and normal rotations on the mid-surface as

$$\begin{Bmatrix} u_z(x,y) \\ \phi_x(x,y) \\ \phi_y(x,y) \end{Bmatrix} \equiv \begin{Bmatrix} U_z(x,y,0) \\ \frac{\partial U_x}{\partial z}(x,y,0) \\ \frac{\partial U_y}{\partial z}(x,y,0) \end{Bmatrix} = \frac{1}{2G}\begin{Bmatrix} \Psi \\ -\frac{\partial}{\partial x}\left[\Psi + \frac{h^2}{4(1-\nu)}\nabla^2\Psi\right] \\ -\frac{\partial}{\partial y}\left[\Psi + \frac{h^2}{4(1-\nu)}\nabla^2\Psi\right] \end{Bmatrix}, \tag{6}$$

respectively. Furthermore, we find the following key relations:

$$\phi_x = -\frac{\partial u_z}{\partial x} - \frac{h^2}{4(1-\nu)}\frac{\partial}{\partial x}\nabla^2 u_z, \quad \phi_y = -\frac{\partial u_z}{\partial y} - \frac{h^2}{4(1-\nu)}\frac{\partial}{\partial y}\nabla^2 u_z, \tag{7}$$

$$\frac{\partial\phi_x}{\partial y} = \frac{\partial\phi_y}{\partial x}, \quad \frac{\partial\phi_x}{\partial x} + \frac{\partial\phi_y}{\partial y} = -\nabla^2 u_z. \tag{8}$$

By using the mid-surface variables (6), the 3-D displacements (2)–(4) can be written as

$$U_x = z\phi_x - \frac{4z^3}{3h^2}\left(\phi_x + \frac{\partial u_z}{\partial x}\right) + \frac{\nu z^3}{6(1-\nu)}\frac{\partial}{\partial x}\left(\frac{\partial\phi_x}{\partial x} + \frac{\partial\phi_y}{\partial y}\right), \tag{9}$$

$$U_y = z\phi_y - \frac{4z^3}{3h^2}\left(\phi_y + \frac{\partial u_z}{\partial y}\right) + \frac{\nu z^3}{6(1-\nu)}\frac{\partial}{\partial y}\left(\frac{\partial\phi_x}{\partial x} + \frac{\partial\phi_y}{\partial y}\right), \tag{10}$$

$$U_z = u_z - \frac{\nu z^2}{2(1-\nu)}\left(\frac{\partial\phi_x}{\partial x} + \frac{\partial\phi_y}{\partial y}\right). \tag{11}$$

The above expressions describe the kinematics of a *Full Interior* plate and are valid for any biharmonic $\Psi(x,y)$. If we neglect the