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# The technique of domain superposition to solve piecewise homogeneous elastic problems

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A R T I C L E I N F O

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A B S T R A C T

The Boundary Element Method, a numerical technique that is not based on domain discretization, faces difficulties to model piecewise homogeneous problems that do not appear in many other applications. Thus, in this paper it is presented an alternative methodology for solution of this kind of problem, previously tested successfully for the Laplace Equation, applied here to static cases of linear elasticity. It is substantially different from the classic sub-region technique, since it is based on the sum of elastic energy retained in each distinct sector. Several examples that include cases with irregular domains are simulated showing the robustness and adequacy of the proposed technique. In the absence of analytical solutions, Finite Element Method solutions are used as reference for error evaluation.

## 1. Introduction

One can say that a most unsuitable application for the Boundary Element Method (BEM) concerns the modeling of problems with non-homogeneous domains, very common case in soil mechanics, geophysics, etc. Usually domain discretization methods are preferred to model this important class of problems, e.g., the Finite Element Method [1,2] the Finite Difference Method [3], etc.

The "sub-region" technique is still the most well known BEM technique to deal with sector located heterogeneity, [4]. As the procedure is based on domain partition, a simple concept, significant changes have not been observed for this technique along the time [5,6]. This does not mean that the procedure is immune to criticism; in certain complex situations the insertion of many internal boundaries produces harmful effects, such as loss of accuracy, increase in computational cost and more elaborate programming.

Concerning fracture mechanics, where BEM is more efficient than other traditional discrete methods, similar restrictions may occur. If layered-materials are involved, the sub-region technique must be introduced and the order of the final system matrix can be increased significantly. Thus, to preserve the advantages of BEM when there is a large number of elements some strategies have been proposed to reduce the size of the final matrix [7]. Anyway, the creation of additional internal boundaries and consequent loss of numerical precision due to the approximation persists.

This work presents the extension of a technique that showed encouraging results solving scalar potential problems [8]. Its idea is substantially different from the classic sub-region technique, since energy principles support the proposed procedure, which hereinafter is named Domain Superposition Technique (DST).

Physically, using the DST the original problem as a whole is modeled by a superposition of a homogeneous background domain and other sub-domains with distinct properties. Elastic energy in each sector is computed suitably generating a consistent mathematical model given in a usual form of BEM integrals. Mathematically, the DST links all sub-domains through influence coefficients, which are given by integrations carried out on the surrounding boundary and all sectors boundaries, with the source points located at all nodal points, either internal or on the boundaries. The final DST H matrix is full, but its order is lower than the order of standard sub-region H matrix, since

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the tractions on internal boundaries do not appear in the final DST system.

It must be highlighted that the same DST idea concerning the energy contribution of each homogeneous sector exists in problems with body forces. Loeffler and Mansur [9] used this approach to account for sectorial loads with the Dual Reciprocity technique [10].

It should be emphasized that the approaches concerning piecewise homogeneous domains, for both scalar and vectorial problems, were studied before using the mathematical formalism of the Potential Theory [11–13]. Nevertheless, comprehensive numerical simulations were not presented in these works.

The matrices assembly technique of the DST procedure is similar to that found in solution of soil-structures interaction problems when FEM and BEM are employed together [14], as well as to cases of integration between zoned plate bending [15,16] in which common nodal points of different systems are assembled in a global matrix. The numerous analysis carried out by Venturini and others [17,18] concerning plate-beam-column integrated systems have induced him to propose models which employed schemes close to the DST procedure [19], however, based exclusively on algebraic manipulations of sub-regions integral equations. Later on, in another work Venturini improved his method [20], still small differences related to the DST remained, although the matrix expressions and the numerical results are very similar. Recently Wagdy and Rashed [21] also proposed a formulation where an additional stiffness matrix is assembled for boundary integral equations in which different sub-domains are connected using internal points.

It must be highlighted that there are meaningful theoretical differences concerning the aforementioned methods and the DST technique: all mentioned works do not employ the concept of energy superposition to connect suitably different sub-domains where a surrounding homogeneous domain is taken as background. Just the idea of linkage of distinct domains using internal points is employed in the aforementioned papers.

The good performance achieved in Laplace problems accredits the DST for other more elaborate applications without great difficulties of implementation, such as elasticity problems, the goal of the present study. The extension of DST to this kind of problem opens an opportunity to examine other interesting applications such as plasticity and fracture mechanics. Despite being a new application, the central idea does not change and its computational advantages remain: the proposed technique is still much easier to implement computationally than the sub-region technique, since all new influence coefficients related to the sectors can be added directly to the classic H BEM matrix.

## 2. The technique of domain superposition

Considering a two-dimensional medium as being continuous, homogeneous, elastic, linear, isotropic, in static conditions, without body forces, the governing differential equation associated to this problem is the Navier's Equation [22]. This equation using indicial notation and the constants of Lamé $\lambda$ and $\mu$ [23] is given by:

$$\mu\, u_{j,ii}(\mathbf{X}) + (\lambda+\mu)\, u_{i,ij}(\mathbf{X}) = 0 \tag{1}$$

In Eq. (1), $u_i(\mathbf{X})$ represents the vector component of the displacement field in "$i$" direction and $\mathbf{X}$ represents a point with coordinates $(x_1, x_2)$. Possible body forces are not considered here.

Suitable mathematical operations are applied, beginning with the integral formulation of the governing equation using the Kelvin fundamental solution $u_j^*(\xi;\mathbf{X})$ [4] as auxiliary function. Thus, the following integral equation may be written on a homogeneous domain $\Omega(\mathbf{X})$

$$\mu \int_{\Omega} u_j(\mathbf{X})_{,ii}\, u_j^*(\xi;\mathbf{X})\, d\Omega(\mathbf{X}) + (\lambda+\mu)\int_{\Omega} u_i(\mathbf{X})_{,ij}\, u_j^*(\xi;\mathbf{X})\, d\Omega(\mathbf{X}) = 0 \tag{2}$$

In order to present the features of the DST, a domain consisting of two regions with distinct physical properties is considered, as shown in Fig. 1, in which the complete domain $\Omega(\mathbf{X})$ is composed of the sum of

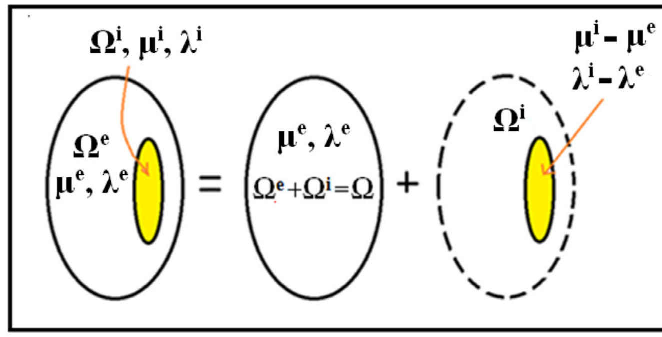


**Fig. 1.** Complete and sectorial domains with homogeneous properties.

$\Omega^e$ and $\Omega^i$; both $\lambda^e$, $\lambda^i$, $\mu^e$ and $\mu^i$ are physical properties, constant inside each sub-domain. In this formulation, unlike what is done in the traditional sub-regions approach, a complete or surrounding domain with homogeneous properties is elected and the other sub-domains are correlated with it.

The constitutive properties are constant inside each sub-domain. Thus, considering that the kernel of the integrals is comprised by integrable functions, the following integral equation can be exposed based on the domain division concept described above:

$$\begin{aligned} &\mu^e \int_{\Omega^e} {u^e}_j(\mathbf{X})_{,ii}\, u_j^*(\xi;\mathbf{X})\, d\Omega^e(\mathbf{X}) + (\lambda^e+\mu^e)\int_{\Omega^e} {u^e}_i(\mathbf{X})_{,ij}\, u_j^*(\xi;\mathbf{X})\, d\Omega^e(\mathbf{X}) \\ &+ \mu^i \int_{\Omega^i} {u^i}_j(\mathbf{X})_{,ii}\, u_j^*(\xi;\mathbf{X})\, d\Omega^i(\mathbf{X}) + (\lambda^i+\mu^i)\int_{\Omega^i} {u^i}_i(\mathbf{X})_{,ij}\, u_j^*(\xi;\mathbf{X})\, d\Omega^i(\mathbf{X}) = 0 \end{aligned} \tag{3}$$

Performing some simple mathematical operations considering that $\lambda^i = \lambda^e + \lambda^*$ and $\mu^i = \mu^e + \mu^*$, the following integral equation is achieved:

$$\begin{aligned} &\mu^e \int_{\Omega} {u^e}_j(\mathbf{X})_{,ii}\, u_j^*(\xi;\mathbf{X})\, d\Omega^e(\mathbf{X}) + (\lambda^e+\mu^e)\int_{\Omega} {u^e}_i(\mathbf{X})_{,ij}\, u_j^*(\xi;\mathbf{X})\, d\Omega^e(\mathbf{X}) \\ &+ \mu^* \int_{\Omega^i} {u^i}_j(\mathbf{X})_{,ii}\, u_j^*(\xi;\mathbf{X})\, d\Omega^i(\mathbf{X}) + (\lambda^*+\mu^*)\int_{\Omega^i} {u^i}_i(\mathbf{X})_{,ij}\, u_j^*(\xi;\mathbf{X})\, d\Omega^i(\mathbf{X}) = 0 \end{aligned} \tag{4}$$

Eq. (4) synthesizes the DST aim: the problem as a whole can be analyzed as a superposition of a contribution related to a background homogeneous domain and other sectors that are also homogeneous. Due to the BEM features, this contribution is given in terms of balance of elastic energy.

## 3. Boundary integral equations

Eq. (4) can be rewritten in terms of the following boundary integrals, after application of the Divergence Theorem [4,24]:

$$\begin{aligned} &\mu^e\left[-P_j u_j(\xi) + \int_{\Gamma} p_j(\mathbf{X}) u_j^*(\xi;X)\, d\Gamma(\mathbf{X}) - \int_{\Gamma} p_j^*(\xi;\mathbf{X}) u_j(\mathbf{X})\, d\Gamma(\mathbf{X})\right] \\ &+ \mu^*\left[-P_j u_j^i(\xi) + \int_{\Gamma^i} p_j^i(\mathbf{X}) u_j^*(\xi;\mathbf{X})\, d\Gamma^i(\mathbf{X}) - \int_{\Gamma^i} p_j^*(\xi;\mathbf{X}) u_j^i(\mathbf{X})\, d\Gamma^i(\mathbf{X})\right] = 0 \end{aligned} \tag{5}$$

For convenience, a dyadic structure is adopted for the fundamental solution and its associated traction derivative, denoted respectively $u_{ij}{}^*$ and $p_{ij}{}^*$,to represent displacements and tractions generated in the direction $j$ at field point $X$, as results of a unit load acting in the direction $i$ applied at source point $\xi$. When a dyadic coefficient $C_{ij}$ is introduced as a function of the position of the source point (if it lies within the domain, outside it or exactly on the boundary), the complete inverse boundary integral equation takes the following form: