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A localized transform-based meshless method for solving time fractional wave-diffusion equation

Marjan Uddin*, Kamran, Amjad Ali

Department of Basics Sciences and Islamiat, University of Engineering and Technology, Peshawar, Pakistan



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ABSTRACT

In the present work, a hybrid transform-based localized meshless method is constructed for the solution of fractional diffusion-wave equations. The time stepping procedure is avoided to overcome the problem of time instability related to meshless methods. The issue of ill conditioning related to meshless differentiation matrices is resolved by incorporating small local system matrices. The time fractional diffusion-wave equation is selected to test the method. A clear improvement is observed in terms of stability, accuracy and ill-conditioning.

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Introduction

Meshless methods have become increasingly popular over the last twenty years [4,6,8,19,25,32]. There are two approaches solving boundary value problems that based on the strong form and the weak form. The strong form seeks to collocate exactly the governing differential or integral equations and boundary conditions on a set of nodes inside the domain as well as on the boundary. For example, the Boundary Element Method (BEM) [6] requires integration and needs boundary elements. While the spectral methods possess spectral convergence, and needs no elements. But these methods require to collocate over a domain of regular shape using a set of structured nodes. On the contrary the radial basis function (RBF) method [22,23] and the method of fundamental solution (MFS) [6] can use scattered (unstructured) nodes, and hence are truly meshless methods. These global collocation methods however have the tradeoffs of high matrix condition number, and that the approximate solution is continuous even if discontinuity is present in the true solution. These shortcomings can be overcome by using the localized meshless methods [32,36].

The second approach is based on the weak form, such as the variational or weighted residual method. The weighted residual is integrated over the domain, and minimized. The finite element method (FEM) [4] subdivides the domain into elements and performs local minimization within each element. The localized scheme results in banded and well-conditioned matrix. However currently further research is needed to handle the issues of error convergence, stability of the matrix, computational efficiency, simplicity of mathematical formulation and numerical implementation, adaptability to parallel processing, and the flexibility to solve challenging problems, such as moving boundary, fragmentation, high-dimensional problems and problems with non-smooth

solutions. As no method can perform the best on all occasions, combined or hybrid methods may offer the best of both worlds.

In the present work, we combined the Laplace transform [28,40,43] with localized RBF method [41,42]. We investigated its applicability for solving time fractional diffusion-wave equations. An improvement against other meshless methods using time stepping procedure for solving the same problem have been observed. The combination of Laplace transform with some other methods have been successfully achieved earlier and is available in the literature. Only small amount of work is available using the coupling of Laplace transform with other methods. For example in the work of [12] the authors coupled the Laplace transform with boundary-particle method for solving time fractional diffusion equation. In [30] the authors have combined the Laplace transform with Kansa method. Similarly the authors [13] studied the combination of Laplace transform with RBF method on unit sphere for solving Heat equation. The combination of Laplace transform with finite element, finite difference method and spectral method can be found in the references [10,21,27,28,35,43]. Here it is attempted to combine the Laplace transform with localized meshless method and apply it to time fractional diffusion-wave equation. Recently, differential equations of fractional order have been widely used in many applications in science and engineering. Many phenomena in chemistry, fluid mechanics, finance, viscoelasticity, physics and other related fields can be successfully described using mathematical tools from fractional calculus [1–3,14–16,18,24,29,44].

1. Analysis of the method

In this section we propose a hybrid meshless method for time fractional linear PDEs. In the proposed method we eliminate the time vari-

* Corresponding author.

E-mail address: marjan@uetpeshawar.edu.pk (M. Uddin).

able by integral transformation and for the time independent PDE localized meshless numerical scheme will be constructed. Our numerical scheme for approximating time fractional differential equation of order $1 < \alpha < 2$ of the form

$$\partial_t^\alpha u(\mathbf{x}, t) + \mathcal{L}u(\mathbf{x}, t) = f(\mathbf{x}, t), \quad \mathbf{x} \in \Omega \subset \mathbb{R}^d, \quad d \geq 1. \quad (1.0.1)$$

with the initial conditions

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad u_t(\mathbf{x}, 0) = u_1(\mathbf{x}), \quad \mathbf{x} \in \Omega, \quad (1.0.2)$$

and the boundary conditions

$$\mathcal{B}u(\mathbf{x}, t) = g_1(t), \quad \mathbf{x} \in \partial\Omega, \quad (1.0.3)$$

where \mathcal{L} is a linear spatial differential operator and \mathcal{B} is boundary operator and ∂_t^α is the Caputo fractional partial derivative of order α defined by

$$\partial_t^\alpha u(\mathbf{x}, t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\partial^2 u(\mathbf{x}, s)}{\partial s^2} \frac{ds}{(t-s)^{\alpha-1}}, \quad \alpha \in (1, 2). \quad (1.0.4)$$

Let the Laplace transform of $v(t)$ be denoted by

$$\mathcal{L}\{v(t)\} = V(z) = \int_0^\infty e^{-zt} v(t) dt, \quad (1.0.5)$$

and the Laplace transform of the Caputo derivative is defined as

$$\mathcal{L}\{\partial_t^\alpha v(t)\} = z^\alpha V(z) - \sum_{i=0}^{p-1} z^{\alpha-i-1} v^{(i)}(0), \quad p-1 < \alpha < p \in \mathbb{Z}^+, \quad (1.0.6)$$

then applying the Laplace transform to Eqs. (1.0.1)–(1.0.3), we get

$$[z^\alpha U(\mathbf{x}, z) - z^{\alpha-1} u_0 - z^{\alpha-2} u_1] + \mathcal{L}\{U(\mathbf{x}, z)\} = F(\mathbf{x}, z), \quad \mathbf{x} \in \Omega, \quad (1.0.7)$$

and

$$\mathcal{B}\{U(\mathbf{x}, z)\} = G_1(z), \quad \mathbf{x} \in \partial\Omega, \quad (1.0.8)$$

respectively. Thus, we have the following system of linear differential equations

$$[z^\alpha I + \mathcal{L}]\{U(\mathbf{x}, z)\} = G(\mathbf{x}, z), \quad \mathbf{x} \in \Omega. \quad (1.0.9)$$

$$\mathcal{B}\{U(\mathbf{x}, z)\} = G_1(z), \quad \mathbf{x} \in \partial\Omega, \quad (1.0.10)$$

where

$$G(\mathbf{x}, z) = z^{\alpha-1} u_0 + z^{\alpha-2} u_1 + F(\mathbf{x}, z),$$

and then the solution $u(\mathbf{x}, t)$ of Problems (1.0.1)–(1.0.3) can be obtained by using inverse Laplace transform

$$u(\mathbf{x}, t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{zt} U(\mathbf{x}, z) dz = \frac{1}{2\pi i} \int_\Gamma e^{zt} U(\mathbf{x}, z) dz, \quad \sigma > \sigma_0, \quad (1.0.11)$$

where Γ is suitable path joining $\sigma - i\infty$ to $\sigma + i\infty$. The solution of Problems (1.0.1)–(1.0.3) is mainly based on approximating the complex contour integral (1.0.11) along the chosen path either parabolic or hyperbolic, etc.

Numerical integration of the integral defined by the Eq. (1.0.11) is hard to compute because of the highly oscillatory exponential factor on the contour of integral and slowly decaying transform $U(\mathbf{x}, z)$, $z = \sigma + iy$, as $|y| \rightarrow \infty$. To circumvent the slow decay of the transform $U(\mathbf{x}, z)$ the strategy of Talbot [37] can be used. He suggested that the Bromwich line $z = \sigma + iy$, $-\infty < y < \infty$ can be transformed to a contour which starts and ends in left half plane, so that at each end $\text{Re}(z) \rightarrow -\infty$, in such a case the integrand in (1.0.11) will decay rapidly because of the exponential factor. And this makes the integral defined by (1.0.11) suitable for approximation by trapezoidal or mid point rule [26]. By Cauchy's theorem such a deformation is allowed if $U(\mathbf{x}, z)$ has no singularities, and $|U(\mathbf{x}, z)| \rightarrow 0$ if $\text{Re}(z) \leq \sigma_0$ as $|z| \rightarrow \infty$. (If $U(\mathbf{x}, z)$ have singularities with unbounded imaginary part, then Talbot method may not work). In our approximation we used two types of contours namely the parabolic as well as hyperbolic, respectively.

In the work of [43] the parabolic path is parameterized as

$$z = \mu(1 + is)^2. \quad (1.0.12)$$

For the strip $s = \gamma + ic$, where $c > 0$, $-\infty < \gamma < \infty$, the parabolic contour reduces to

$$z(\gamma) = \mu((1-c)^2 - \gamma^2) + 2i\mu\gamma(1-c). \quad (C_1) \quad (1.0.13)$$

Where μ is a parameter which controls width of the given contour. For best approximation we need to find optimal contour of integration. Weideman and Trefethen [43] have optimized the parabolic contour and obtained the optimal values of the parameters which guarantee to maintain small error in the time interval $t_0 < t < T$. For given t_0 , T , t and M , the optimal parameters are given by

$$\mu = \frac{\pi}{4\sqrt{8\Lambda+1}} \frac{M}{t}, \quad k = \frac{\sqrt{8\Lambda+1}}{M}, \quad \Lambda = \frac{T}{t_0},$$

and the corresponding error estimate is of the order

$$E_1 = |u_k(\mathbf{x}, t) - u(\mathbf{x}, t)| = O\left[e^{-\left(\frac{2\pi}{\sqrt{8\Lambda+1}}\right)^M}\right], \quad M \rightarrow \infty \quad (1.0.14)$$

In our computations, M is defined by $m = 2M/k$, where m is the number of quadrature points along the contour of integration and k denote the step size of the trapezoidal rule.

The hyperbolic path is parameterized in form see for example [28],

$$z(\gamma) = \omega + \lambda(1 - \sin(\delta - i\gamma)), \quad \text{for } -\infty < \gamma < \infty, \quad (C_2) \quad (1.0.15)$$

with $\lambda > 0$, $\omega \geq 0$, $0 < \delta < \beta - \frac{1}{2}\pi$, and $\frac{1}{2}\pi < \beta < \pi$ (for detail see [28]). The authors derived the optimal parameters for hyperbolic contour and its corresponding error estimate given by

$$E_2 = |u_k(\mathbf{x}, t) - u(\mathbf{x}, t)| = O(l(\rho_r M) e^{-\mu M}), \quad (1.0.16)$$

where, $l(x) = \max(1, \log(1/x))$, $r > 0$, $\mu = (1 - \theta)\bar{r}/b$, $\bar{r} = 2\pi r$, $0 < \theta < 1$, $\rho_r = \theta\bar{r}\tau \sin(\delta - r)/b$, $b = \cosh^{-1}(1/(\theta\tau \sin(\delta)))$, $\tau = t_0/T$, $t_0 \leq t \leq T$, $0 < t_0 < T$, $\lambda = \theta\bar{r}M/(bT)$, and $k = b/M$.

The numerical approximation of the integral representation (1.0.11) involves the following steps:

- Select m points $\{z_1, z_2, \dots, z_m\}$ along the path of integration (e.g parabolic or hyperbolic paths) and find the solution $U(\mathbf{x}, z_j)$ of the Eqs. ((1.0.9),(1.0.10)).
- The approximation of (1.0.11) in the complex plane can be obtained by trapezoidal rule with uniform step size k , and m quadrature points $z_j = z(\gamma)$, where $\gamma = -M : k : M$, and $\mathbf{z}(\gamma)$ can be obtained from (1.0.13) to (1.0.15) for parabolic and hyperbolic path, respectively.

$$u_k(\mathbf{x}, t) = \frac{k}{2\pi i} \sum_{j=-M}^M e^{z_j t} U(\mathbf{x}, z_j) \dot{z}_j. \quad (1.0.17)$$

2. Stability

To discuss the stability of systems ((1.0.9),(1.0.10)), in discrete form this system can be represented as

$$WU = \mathbf{b}, \quad (2.0.1)$$

where W is $N \times N$ sparse differentiation matrix which can be obtained by localized kernel based method discussed in Section 3. The stability constant corresponding to system (2.0.1) is given by

$$C = \sup_{U \neq 0} \frac{\|U\|}{\|WU\|}. \quad (2.0.2)$$

The value of C is finite using any type of discrete norms $\|\cdot\|$ on \mathbb{R}^N . The above equation can be expressed as

$$\|W\|^{-1} \leq \frac{\|U\|}{\|WU\|} \leq C, \quad (2.0.3)$$

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