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New preconditioning techniques for the steady and unsteady buoyancy driven flow problems

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A B S T R A C T

In this paper we study the performances of generalized minimal residual method (GMRES) preconditioned with geometric multigrid (GMG), applied to steady and unsteady buoyancy driven flow problems, discretized with the finite element method. For the unsteady case, the second order Crank–Nicolson method is used for the temporal discretization. At each geometric multigrid level, we use Richardson iterative solvers preconditioned with different combinations of physics-based and domain decomposition preconditioners. Three different preconditioners are considered: incomplete LU decomposition (ILU), overlapping Vanka-type domain decomposition for additive Schwarz method (ASM), and field split (FS) physics-based decomposition. We also analyze the effect on the smoother of how the variables are ordered, and in particular whether the leading variable is the velocity or the temperature, resulting in six classes of preconditioners: ILU_VT, ILU_TV, ASM_VT, ASM_TV, FS_VT and FS_TV. The eigenvalue analysis for the six preconditioners is conducted to study the rate of GMRES convergence under several Prandtl numbers. The numerical performances of nested combinations of the above preconditioners are compared. Numerical results show that the pair of FS_VT and FS_TV preconditioners works better than the other two pairs, and that the FS_TV preconditioner always performs the best in terms of the computational time for all the steady and unsteady cases.

1. Introduction

Practical applications of buoyancy driven flow problems can be found in both nature and industry. Examples for this type of flow are atmospheric fronts, katabatic winds, boiling water nuclear reactors, crystal growth, and electronic cooling [1–7]. In numerical linear algebra the discretization of this type of problems is attractive, since it naturally leads to systems of nonlinear algebraic equations which exhibit clear block structure [8], and for such systems new block preconditioners can be designed and analyzed.

The generalized minimal residual method (GMRES) preconditioned with the geometric multigrid (GMG) is of great interest for these studies. This solver has been used to solve various partial differential equation (PDE) problems. Hager and Lee [9] found that the GMG preconditioned GMRES was an efficient scheme for solving the unsteady Euler equations. Oosterlee and Washio [10] proposed three types of GMG methods to solve Poisson-type equations and convection–diffusion equations with dominating convection term. In their work they found that the GMG used as a preconditioner for GMRES was much more robust and efficient than using it as a solver. Wright et al. [11] used the GMG preconditioned GMRES to solve two-

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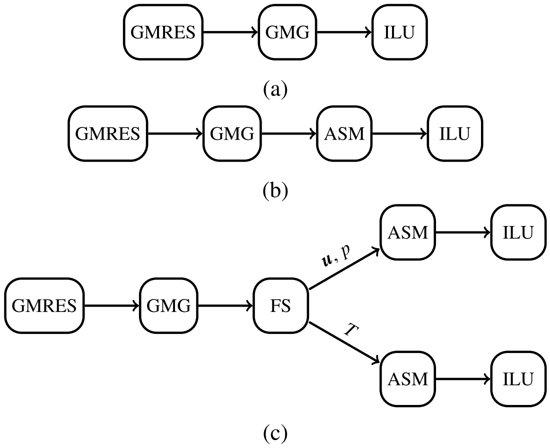

Fig. 1. The combinations of preconditioners (a) ILU (b) AMS (c) FS.

phase Gel dynamics problems. And again it was showed to be an efficient and robust algorithm for solving the momentum and incompressible continuity equations. In his dissertation Gatsis [12] extended this solver to compressible Navier–Stokes equations. It was found the GMG preconditioned GMRES performed effectively in most cases in terms of the number of GMRES per Newton iterations. More examples of GMG preconditioned GMRES can also be found in the his dissertation references. Most recently, Bohowmik [13] applied this algorithm to reaction–diffusion systems. It was found that the GMG preconditioned GMRES solver for these problems converged much faster than the unpreconditioned GMRES solver.

In this paper, the GMG preconditioned GMRES solver is extended to solve buoyancy driven flow problems. Each GMG level sub-solvers can be further preconditioned with other preconditioners. The preconditioners that we develop and compare are either physics-based or domain decomposition preconditioners. The purpose of this comparison is to highlight how the different way in which blocks are extracted from the original Jacobian matrix affects the numerical performance of the GMG preconditioned GMRES solver.

The fluid variables (velocity and pressure) and the thermal variables (temperature) are defined all over the physical domain. This yields a clear block structure associated to the PDE system. Preconditioners constructed using these blocks are referred to as *physics-based* preconditioners [14,15,8]. On the other hand, the computational domain can be divided into smaller subdomains, overlapping or non overlapping, and on each of them a small algebraic sub-system with the same block structure of the original system can be extracted and solved only for the fluid and the temperature variables associated to the subdomain. Preconditioners based on the domain splitting are called *domain decomposition*. The Jacobian matrix associated with the buoyancy driven flow problem has a saddle point sub-block structure due to the incompressibility constraint of the fluid flow. These kinds of problems require a particular choice of overlapping domain decomposition strategy that was first observed by Vanka in his original work [16], and followed in many other works, see for example [17–21].

There are several combinations how these classes of preconditioners can be used together, generating a cascading hierarchy of preconditioning strategies. In this work we implement, analyze and compare the following three different preconditioners applied to a Richardson iterative solver,

1. Incomplete LU decomposition preconditioner (ILU).
2. Additive Schwarz Method (ASM), where the domain subdivision is given by overlapping Vanka-type decomposition, and where each resulting sub-block is preconditioned with ILU as before.
3. Field split (FS) physics-based preconditioner, with sub-systems preconditioned with ASM as before.

A schematic of the different preconditioners is given in Fig. 1.

We also analyze the effect of reordering the variables in the Jacobian matrix in terms of computational time. In particular we study the effects of using the temperature or the velocity field as the leading variable in the Jacobian matrix. This counts for three different pairs of preconditioners: ILU_VT, ILU_TV, ASM_VT, ASM_TV, FS_VT and FS_TV.

The difference between the proposed preconditioner classes lies basically in the way in which the equations are ordered and in the way the blocks are extracted from the original system matrix in order to construct the preconditioner. While some of the preconditioners used are fairly standard if taken separately, it is the first time that a physic-based preconditioner (field-split) is combined with domain-decomposition preconditioner (ASM/Vanka) to solve coupled problems such as fluid-thermal interaction (FTI) problems. Every time a new preconditioner (and corresponding solver) is added into the preconditioner cascade, new direct and inverse mappings between the variables of two consecutive levels have to be built. This tasks become cumbersome when the next level is an ASM/Vanka preconditioners. Moreover the inclusion of this type of preconditioners inside a geometric multigrid preconditioner makes the task threefold challenging.