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Two-level schemes for the advection equation

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ABSTRACT

The advection equation is the basis for mathematical models of continuum mechanics. In the approximate solution of nonstationary problems it is necessary to inherit main properties of the conservatism and monotonicity of the solution. In this paper, the advection equation is written in the symmetric form, where the advection operator is the half-sum of advection operators in conservative (divergent) and non-conservative (characteristic) forms. The advection operator is skew-symmetric. Standard finite element approximations in space are used. The standard explicit two-level scheme for the advection equation is absolutely unstable. New conditionally stable regularized schemes are constructed, on the basis of the general theory of stability (well-posedness) of operator-difference schemes, the stability conditions of the explicit Lax–Wendroff scheme are established. Unconditionally stable and conservative schemes are implicit schemes of the second (Crank–Nicolson scheme) and fourth order. The conditionally stable implicit Lax–Wendroff scheme is constructed. The accuracy of the investigated explicit and implicit two-level schemes for an approximate solution of the advection equation is illustrated by the numerical results of a model two-dimensional problem.

1. Introduction

Mathematical models of continuum mechanics [1,2] describe the transport of scalar and vector quantities due to advection. In particular, the basic equation of hydrodynamics is the continuity equation. Advective transfer causes the fulfillment of conservation laws [3,4]. In addition, there are properties of the positivity and monotonicity of the solution. Such important properties of the differential problem must be inherited when passing to the discrete problem [5,6].

For spatially approximation, conservative approximations are constructed on basis of using the conservative (divergent) form of the advection equation. Most naturally such technology is implemented when using the integro-interpolation method (balance method) on regular and irregular grids [7], in the control volume method [4,8]. The construction of monotonic approximations is discussed in many papers (see, for example, [9–11]). In [12,13] standard linear approximations are considered for convection–diffusion problems.

Currently, the main computing technology for solving applied problems is the finite element method [14,15]. It is widely used in computational fluid dynamics [16,17]. Monotonization of the solution is achieved by using various linear and nonlinear variants of stabilization techniques. It should be noted that the standard formulation of the equations of continuous

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medium mechanics in the conservative or non-conservative form is poorly suited for applying finite element approximations, for which the Hilbert spaces are natural.

Separate attention deserves the problems of constructing and investigating approximations in time. When solving boundary value problems for partial differential equations, two-level schemes (θ-method, schemes with weights) are traditionally widely used [10,18,19]. Research of this schemes can be based on the general theory of stability (well-posedness) of operator-difference schemes [7,20]. In particular, unimprovable (coinciding necessary and sufficient) stability conditions can be used, which are formulated as operator inequalities in a finite-dimensional Hilbert space. The achieved level of theoretical stability research allows us to abandon the heuristic methods widely used in computational fluid dynamics to study the stability of difference schemes: the von Neumann method for stability analysis, Fourier analysis, the principle of frozen coefficients, the consideration of the problem without taking into account the boundary conditions.

In this paper, the standard finite-element approximation in space is used for the nonstationary equation. The advection equation is written in the so-called symmetric form [13], when the advection operator is the half-sum of the advection operators in the conservative (divergent) and non-conservative (characteristic) forms. Thus the continuum mechanics equations are written using the SD (Square root from Density) variables [21,22]. In this case the corresponding conservation laws are a direct consequence of the skew-symmetry of the advection operator. The conservativeness property is related to the preservation of the norm of the solution of the non-stationary advection equation, with stability with respect to the initial data. This property takes place not only for the solution, but also for some of its transformations. In this case, we are talking about the property of multiconservativeness. The principal point is related with the fact that the most important skew-symmetry property of the advection operator is inherited for finite element approximations.

For the advection equation an explicit two-level scheme as well as all schemes with a weight lower than 0.5, are absolutely unstable. At the same time, extensive computational practice aims us to use the explicit schemes in these problems. In such schemes, the conditional stability (the Courant–Friedrichs–Lewy (CFL) condition) with a time-step limited by the Courant number is provided in fact by refusing the skew-symmetry of the advection operator — the use of dissipative approximations. In addition, the standard explicit schemes are not conservative.

We construct conditionally stable schemes for the advection equation based on the principle of regularization of operator-difference schemes [23,24], when stability is provided by a small perturbation. The explicit second-order Runge–Kutta scheme [25,26] is considered. In this context, the classical explicit Lax–Wendroff scheme [27] is also considered as regularization of the second-order Runge–Kutta scheme — the perturbation of the advection operator squared. Using the stability criteria of two-level operator-difference schemes, the stability conditions of regularized schemes and the explicit Lax–Wendroff scheme are obtained. Effective computational implementation of explicit schemes using finite element approximation in space is provided by using the diagonalization procedures of the mass matrix (mass-lumping procedure) [28,29].

Of greatest interest are implicit two-level schemes for the advection equation, which belong to the unconditionally stable class. The classical Crank–Nicolson scheme has a second-order of accuracy, is unconditionally stable and multiconservative. For the advection problems under the consideration, we can use a scheme of the fourth-order of accuracy, which is also unconditionally stable and multiconservative. A certain drawback of this scheme is associated with the need to use the lumping procedure. An implicit version of the Lax–Wendroff scheme is proposed, which is conditionally stable, but has a higher accuracy than the explicit Lax–Wendroff scheme and does not require the diagonalization of the mass matrix.

The paper is organized as follows. A model two-dimensional problem for the advection equation is formulated in Section 2. Approximation in space is constructed using Lagrangian finite elements, the main properties of the problem solution are noted. In Section 3, we consider known and new explicit difference schemes for the advection equations, and investigate the stability conditions. Central for this work is Section 4. Implicit schemes, their stability and conservatism are studied here. In Section 5, numerical experiments on the accuracy of explicit and implicit schemes are discussed for the model IBV problem. The results of the work are summarized in Section 6.

2. Problem statement

In a bounded two-dimensional domain Ω, we consider the advection equation written in the symmetric form. A standard finite element approximation in space is used. The problem of constructing approximations in time is formulated in such a way that the approximate solution inherits the basic properties of the solution of the differential problem.

2.1. Differential problem

The Cauchy problem is considered in the domain Ω ($\boldsymbol{x} = (x_1, x_2) \in \Omega$)

$$\frac{dw}{dt} + \mathcal{A}w = 0, \quad 0 < t \le T, \tag{1}$$

$$w(0) = w^0, \tag{2}$$

using notation $w(t) = w(\boldsymbol{x}, t)$. The operator of advection (convective transport) $\mathcal{A}$ is assumed to be constant and is written in the symmetric form: