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Spectral element computation of high-frequency leaky modes in three-dimensional solid waveguides

F. Treyssède

LUNAM Université, IFSTTAR, GERS, GeoEND, F-44340 Bouguenais, France

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ABSTRACT

A numerical method is proposed to compute high-frequency low-leakage modes in structural waveguides surrounded by infinite solid media. In order to model arbitrary shape structures, a waveguide formulation is used, which consists of applying to the elastodynamic equilibrium equations a space Fourier transform along the waveguide axis and then a discretization method to the cross-section coordinates. However several numerical issues must be faced related to the unbounded nature of the cross-section, the number of degrees of freedom required to achieve an acceptable error in the high-frequency regime as well as the number of modes to compute. In this paper, these issues are circumvented by applying perfectly matched layers (PML) along the cross-section directions, a high-order spectral element method for the discretization of the cross-section, and an eigensolver shift suited for the computation of low-leakage modes. First, computations are performed for an embedded cylindrical bar, for which literature results are available. The proposed PML waveguide formulation yields good agreement with literature results, even in the case of weak impedance contrast. Its performance with high-order spectral elements is assessed in terms of convergence and accuracy and compared to traditional low-order finite elements. Then, computations are performed for an embedded square bar. Dispersion curves exhibit strong similarities with cylinders. These results show that the properties of low-leakage modes observed in cylindrical bars can also occur in other types of geometry.

1. Introduction

The modeling of structural waveguides is essential to understand the multimodal and dispersive propagation of guided waves in solids. A typical application of elastic guided waves is non-destructive evaluation (NDE) methods. The knowledge of modal properties of guided waves is necessary to optimize inspection systems by selecting the most suitable propagation modes for NDE, that is to say, the less dispersive and less attenuated modes. For canonical geometries (plates and cylinders), waveguides can be modeled thanks to analytical methods based on the Thomson–Haskell method or the global matrix method [1,2] to provide modal characteristics as a function of frequency (dispersion curves).

Waveguides can be classified into two categories: *closed* waveguides (of bounded cross-section) and *open* waveguides (of unbounded cross-section). Of particular interest in this paper is the latter category, which includes: buried or embedded structures, partially buried waveguides and surface waveguides. Open waveguides are widely encountered in civil engineering. As typical applications, one can cite the NDE of steel rock bolts and bridge cables [3,4] or buried pipes. Such structures

E-mail address: fabien.treyssede@ifsttar.fr.

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are elongated and buried into an external medium (*e.g.* grout, concrete, soil), which can usually be considered as unbounded in the transverse direction.

Numerical methods are required for the modeling of arbitrary shape waveguides. One powerful numerical method consists of assuming the axial dependence of acoustic fields in the equations to be solved, and then to apply a discretization method (usually the finite element method (FEM)) to the remaining coordinates, *i.e.* the cross-section.

As far as closed elastic waveguides are concerned, this method has been widely applied under various names, among which: semi-analytical finite element (SAFE) method [5–7], waveguide finite element method [8,9], strip element method [10] or more recently the so-called scaled boundary finite element method (SBFEM) [11,12]. These names indeed all refer to the same theoretical formulation, which will be referred to as *waveguide formulation* throughout this paper. The SBFEM uses a somehow different formalism to derive the equations but yields the same quadratic eigenproblem and elementary matrices as the other methods (the SBFEM actually differs only in the interpolation functions and the eigensystem linear form chosen to solve the problem).

With open waveguides, numerical methods must yet face difficulties due to the unbounded nature of the cross-section. This issue is enhanced by the intrinsic transverse growth to infinity of leaky modes, well-known in electromagnetism (see Ref. [13] for instance). To circumvent these difficulties, the waveguide formulation must be combined with other techniques.

An additional numerical difficulty is that the less attenuated modes in open elastic waveguides usually occur in a very high frequency regime [3,4]. This phenomenon is due to radiation losses, greater in a low-frequency regime, as opposed to material viscoelastic losses (which increases with frequency). Since the modal density increases with frequency, a large number of modes should be computed to identify these high-frequency low-leakage modes. Moreover their computation with the traditional FEM, using low-order elements, would require a very fine discretization involving a huge number of degrees of freedom (dofs).

For modeling open waveguides, one of the simplest numerical procedures consists of creating artificial viscoelastic layers in the surrounding medium for absorbing waves [14,15]. Waveguide formulations combined with the boundary element method have been recently developed to model three-dimensional elastic waveguides embedded in a solid [16,17] or in a fluid [18]. The boundary element method avoids the discretization of the unbounded medium but yields a highly nonlinear eigenproblem, quite difficult to solve. Gravenkamp et al. [19] have proposed a simplified boundary condition, namely the dashpot boundary condition. This dashpot condition is usually no longer accurate for low frequency or for a low contrast of acoustic impedance. Similarly to the boundary element method, Hayashi et al. [20] have applied an exact radiation condition for two-dimensional plates and have successfully transformed the resulting nonlinear eigenproblem into a linear one. This transformation is yet only applicable to fluid external media. Besides, the extension to three-dimensional waveguides requires far-field approximations [21].

An alternative technique consists of using perfectly matched layers (PML). In addition to providing a linear eigenproblem in a straightforward manner, the PML approach remains applicable even for low contrast media. This technique has been applied to electromagnetic waveguides [22,23], scalar waveguide problems [24,25] and more recently to elastic waveguides [26,27]. From a mathematical point of view, this approach turns out to be relevant because both leaky modes and PML can be defined through analytic continuations. Moreover, the convergence of the method for computing resonances in open systems has been proved for scalar wave problems [28,29]. In practice, the user can adjust PML parameters (thickness for instance) to check convergence towards physical solutions. Conversely as far as the dashpot boundary condition and artificial viscoelastic layers are concerned, theoretical explanations are still missing about the expected accuracy or convergence of solutions, in particular when the contrast of impedance becomes weak [30] (a comparison with these methods, although simple to implement, is left beyond the scope of this paper). One drawback with PML is that part of the exterior domain needs to be meshed, which may increase the computational cost compared to boundary element methods. Nevertheless, the thickness of the absorbing region can be greatly reduced compared to viscoelastic layers [27] thanks to the perfectly matched property.

In fact, the main drawback of the PML waveguide formulation is that the absorbing region leads to the existence of modes resonating mainly inside the PML. These modes can be viewed as discrete sets of two radiation mode continua in elastic problems [26]. In practice, solving the eigenproblem can lead to the computation of many radiation modes, of less interest for NDE applications, for only a limited number of leaky modes. And the computation cost of eigensolvers particularly increases with the number of modes to calculate.

The goal of this paper is to propose a numerical method for the computation of high-frequency leaky modes in open solid waveguides. To do so, the waveguide formulation is combined with a PML technique. The resulting approach leads to a nonstandard eigenvalue problem which is quadratic and not self-adjoint. The cross-section is discretized with high-order spectral elements. Their performance for solving this specific eigenproblem is assessed in terms of convergence and accuracy and compared to traditional low-order finite elements. As expected, using high-order spectral elements enables to significantly reduce the number of dofs needed in the high-frequency regime to achieve a given discretization error. To circumvent the problem of calculating many modes, a property specific to low attenuation modes in embedded waveguides is used. In cylindrical structures, it has been found that low-leakage modes are of compressional type with phase velocities close to the longitudinal bulk velocity of the core [3]. The numerical results obtained in the present paper show that this property still holds for other types of geometry. The computation of eigenvalues is hence centered around the wavenumber of bulk longitudinal waves, which allows to drastically reduce the number of modes to compute.