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An efficient time advancing strategy for energy-preserving simulations

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ABSTRACT

Energy-conserving numerical methods are widely employed within the broad area of convection-dominated systems. Semi-discrete conservation of energy is usually obtained by adopting the so-called skew-symmetric splitting of the non-linear convective term, defined as a suitable average of the divergence and advective forms. Although generally allowing global conservation of kinetic energy, it has the drawback of being roughly twice as expensive as standard divergence or advective forms alone. In this paper, a general theoretical framework has been developed to derive an efficient time-advancement strategy in the context of explicit Runge–Kutta schemes. The novel technique retains the conservation properties of skew-symmetric-based discretizations at a reduced computational cost. It is found that optimal energy conservation can be achieved by properly constructed Runge–Kutta methods in which only divergence and advective forms for the convective term are used. As a consequence, a considerable improvement in computational efficiency over existing practices is achieved. The overall procedure has proved to be able to produce new schemes with a specified order of accuracy on both solution and energy. The effectiveness of the method as well as the asymptotic behavior of the schemes is demonstrated by numerical simulation of Burgers' equation.

1. Introduction

Over the past few decades, substantial research efforts have been devoted to the construction of numerical methods mimicking fundamental properties of the underlying mathematical/physical system. These so-called *physics-compatible* discretizations have gained increasing popularity over the years, especially in numerical simulations of turbulent flows [1]. In this last context, energy-preserving numerical methods are usually the preferred choice, as they provide a natural stability bound over long-time integration. Moreover, being free of numerical diffusion, they ensure that the energy cascade is not artificially contaminated, in fully-resolved computations, and that the contribution of subgrid-scale motions is entirely modeled, in under-resolved cases [2].

Finite difference and spectral energy-conserving schemes found in literature are often built upon the skew-symmetric splitting of the convective term, which is defined as an average of the divergence and the advective forms [3,4]. When coupled to operators satisfying a discrete summation-by-parts rule (i.e., centered finite-difference and spectral schemes), skew-symmetric methods ensure semidiscrete global conservation of energy for incompressible flows in the inviscid limit,

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and prevent spurious production or dissipation of kinetic energy by convection for compressible flows [5]. The skew-symmetric splitting yielded relatively stable simulations for both incompressible [6] and compressible [7] turbulence, and proved to be also beneficial in reducing the amplitude of aliasing errors [8].

Despite its remarkable features, one relevant drawback of the skew-symmetric form is that its computation is roughly twice as expensive as standard divergence or advective forms alone [9]. Moreover, full energy conservation (i.e., in time as well as in space) requires the use of costly implicit time-stepping methods, otherwise a (typically dissipative) error is introduced regardless of the spatial scheme. Finally, fully-conservative methods are not always advantageous as they can display aliasing issues for under-resolved computations without numerical regularization [10]. A trade-off between cost-effectiveness and conservation properties is thus warranted.

The additional expense related to the use of the skew-symmetric form has been mentioned by many authors (e.g. [11]). Most of the attempts appeared so far to achieve cost-effective implementations are simply based on using the advective and divergence form at alternate time steps [9,12]. In this paper, a novel time-advancement strategy that mimics the conservation properties of skew-symmetric-based schemes at a reduced computational cost is presented. It is found that optimal energy-conservation properties can be achieved by properly constructed Runge–Kutta schemes in which a different form (advective or divergence) for the convective term is adopted at each stage. This splitting strategy is able to reproduce the effects of the skew-symmetric form on energy conservation, up to a specific order of accuracy. The main achievement is that, since the method is based only on advective and divergence forms, it can be considerably faster than skew-symmetric-based techniques.

The analysis is conducted by considering the inviscid Burgers' equation, which is a well-known prototype equation of considerable physical and mathematical interest. The reason for this choice stems from the fact that, while the general idea can be extended to more complete models, the detailed analysis on convergence and orders of accuracy requires some analytical developments which are, in the first instance, more neatly conducted on a model equation. In this respect, the Burgers' equation can be considered as the simplest partial differential equation reproducing some of the peculiar features of nonlinear convective transport terms present in more realistic models (e.g., the Navier–Stokes equations). Hence, besides the intrinsic value of the application of the present theory to Burgers' equation, the present analysis is intended also as a first step towards the application to more complex systems.

The paper is organized as follows. In Section 2, the discrete energy conservation properties of both spatial and temporal discretizations are reviewed. In Section 3, a first, simple approach to obtain a cost-effective energy-conserving method is analyzed. The theoretical development for energy-preserving Runge–Kutta schemes is presented in Section 4. Results are shown in Section 5. In Section 6, some indications are given for the extension of the main idea to the incompressible Navier–Stokes equations. Finally, Section 7 presents a summary.

2. Conservation properties of semi-discretized equations

2.1. Spatial discretization

Energy-conservation properties of spatial discretizations of nonlinear convective terms can be investigated by considering the inviscid Burgers' equation, which can be formally written as

$$\partial_t u + \mathcal{N}(u) = 0, \tag{1}$$

where the non-linear convective term $\mathcal{N}(u)$ can be expressed in one of the equivalent forms $u\partial_x u$ (*advective*), $\partial_x u^2/2$ (*divergence*) or $(\partial_x u^2 + u\partial_x u)/3$ (*skew-symmetric*). When posed on an interval $[a,b]$ with periodic boundary conditions, Eq. (1) has solutions for which all the moments are conserved:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \frac{u^n}{n}\,dx = \int_a^b u^{n-1}\frac{\partial u}{\partial t}\,dx = -\int_a^b u^n \frac{\partial u}{\partial x}\,dx = -\int_{u(a)}^{u(b)} u^n\,du = 0.$$

Specifically, the total momentum and the total energy of the solution (which are obtained in the cases $n = 1$ and $n = 2$, respectively) remain fixed to their initial values during time evolution.

Although in the continuous case the different forms of the convective term are mathematically equivalent, their spatially discretized counterparts can behave very differently, especially in terms of energy-preserving features and aliasing errors. This is due to the fact that discrete operators generally are not guaranteed to correctly reproduce the numerical equivalents of integration by parts and differentiation chain rule (cf. [4]). The beneficial conservation and aliasing properties of the skew-symmetric form have long been recognized by many authors [4,9,13], while there is much more debate about the other two formulations. The present analysis examines the energy-conserving behavior of the spatially discretized version of Eq. (1) when the different forms are employed.

To pursue the scope, the semi-discretized version of the Burgers' equation is considered, which can be expressed by introducing the matrix $\mathbf{C}(\mathbf{u})$ as

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \mathbf{C}(\mathbf{u})\mathbf{u} = 0. \tag{2}$$