Journal of Computational Physics ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

A projection hybrid finite volume/element method for low-Mach number flows

A. Bermúdez [a], J.L. Ferrín [a], L. Saavedra [b], M.E. Vázquez-Cendón [a,*]

[a] Departamento de Matemática Aplicada, Universidad de Santiago de Compostela, Facultad de Matemáticas, 15782 Santiago de Compostela, Spain
[b] Departamento Fundamentos Matemáticos, Universidad Politécnica de Madrid, E.T.S.I. Aeronáuticos, 28040 Madrid, Spain

ARTICLE INFO

Article history:
Received 21 May 2013
Received in revised form 17 September 2013
Accepted 19 September 2013
Available online xxxx

Keywords:
Low-Mach number flows
Projection method
Finite volume method
Finite element method

ABSTRACT

The purpose of this article is to introduce a projection hybrid finite volume/element method for low-Mach number flows of viscous or inviscid fluids. Starting with a 3D tetrahedral finite element mesh of the computational domain, the equation of the transport-diffusion stage is discretized by a finite volume method associated with a dual mesh where the nodes of the volumes are the barycenters of the faces of the initial tetrahedra. The transport-diffusion stage is explicit. Upwinding of convective terms is done by classical Riemann solvers as the Q-scheme of van Leer or the Rusanov scheme. Concerning the projection stage, the pressure correction is computed by a piecewise linear finite element method associated with the initial tetrahedral mesh. Passing the information from one stage to the other is carefully made in order to get a stable global scheme. Numerical results for several test examples aiming at evaluating the convergence properties of the method are shown.

1. Introduction

Solving the incompressible Navier–Stokes equations by mixed (velocity–pressure) finite element methods is nowadays a very well established subject. The main problem comes from the fact that, for stability purposes, the discretization spaces for velocity and pressure cannot be chosen independently as they have to satisfy the so-called *inf–sup condition* see, for instance, [9]. An alternative approach consists in adding stabilizing terms to the weak formulation (see, for instance, [4] or [11]).

On the other hand, finite volume methods combined with approximate Riemann solvers have been successfully developed for compressible flows in the 1980's (see, [21] and the references therein). The difference compressible/incompressible deeply changes the physics and also the numerics. While for compressible flows pressure is directly related to other flow variables as density and energy via a state equation, for incompressible flows pressure is a Lagrange multiplier that adapts itself to ensure that the velocity satisfies the incompressibility condition. In order to handle the latter situation, the typical explicit stage of finite volume methods has to be complemented with the so-called projection stage where a pressure correction is computed in order to get a divergence-free velocity. Many papers exist in the literature devoted to introduce and analyze projection finite volume methods for incompressible Navier–Stokes equations (see, for instance, [1] or [18]). In order to get stability, staggered grids have to be used to discretize velocity and pressure. While this can be done straightforwardly

* Corresponding author.
E-mail addresses: alfredo.bermudez@usc.es (A. Bermúdez), joseluis.ferrin@usc.es (J.L. Ferrín), laura.saavedra@upm.es (L. Saavedra), elena.vazquez.cendon@usc.es (M.E. Vázquez-Cendón).
URL: http://www.usc.es/ingmat/ (A. Bermúdez).

0021-9991/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jcp.2013.09.029

in the context of structured meshes, the adaptation to unstructured meshes is more tricky (see [2,7,8,19,22]). Other possibility is to extend the SIMPLE method (introduced in [17]), by using multiple pressure variables to account for different physical effects, as was done in [14–16].

On the other hand, projection methods have been also used in combination with finite element discretizations (see [10]). The projection methods, initially introduced for incompressible flows can be easily adapted to low-Mach number flows. In fact, the main difference is that the divergence-free condition for the velocity is replaced by an equation prescribing the divergence of the linear momentum density which is a conservative variable.

The purpose of this paper is to introduce a projection method for low-Mach number flows, both for viscous and inviscid fluids. Starting with a 3D tetrahedral finite element mesh of the computational domain, the equation of the transport-diffusion stage is discretized by a finite volume method associated with a dual finite volume mesh where the nodes of the volumes are the barycenter of the faces of the initial tetrahedra. These volumes have been already used for the 2D shallow water equation (see [2]) and allow us for an easy implementation of flux boundary conditions. For time discretization we use the explicit Euler scheme. Upwinding the convective term is done by classical Riemann solvers as the Q-scheme of van Leer or the Rusanov scheme (see, for instance, [21]). Concerning the projection stage, the pressure correction is computed by continuous piecewise linear finite elements associated with the initial tetrahedral mesh. The use of the above "staggered" meshes together with a simple specific way of passing the information from the transport-diffusion stage to the projection one and vice versa lead to a stable scheme. The former is done by redefining the conservative variable (i.e. the momentum density) constant per tetrahedron. Conversely, the finite element pressure correction is redefined to constant values on the faces of the finite volumes and then used in the transport-diffusion stage. Moreover, several internal iterations can be done at each time step but this is by no means necessary.

The outline of the paper is as follows. In Section 2 the low-Mach number flow equations are recalled. Section 3 is devoted to the transport-diffusion stage. Besides a detailed description of the finite volume discretization we analyze the approximation of the viscous term. For steady-state solutions, we decompose this term into two parts: one is handled implicitly and the other one is taken explicitly in order to avoid solving linear systems at each time step while keeping the CFL condition as the only condition to get stability. In Section 3 we describe the projection stage which is solved by continuous piecewise linear finite elements. The overall algorithm is summarized in Section 4, where we comment on some implementation issues. Finally, in Section 5 we show some numerical results obtained with the developed computer code in order to assess the performance of the method. We include some academic problems in order to analyze the order of convergence as well as several classical test problems from fluid mechanics.

2. Low-Mach number equations

The system of equations described in this section corresponds to a model for low-Mach number flows. The underlying assumption is that the Mach number M is sufficiently small so that pressure p can be written as the sum of a spatially constant function $\overline{\pi}$ and a small perturbation π,

$$p(x,y,z,t)=\overline{\pi}(t)+\pi(x,y,z,t),\qquad \frac{\pi}{\overline{\pi}}=O\left(M^{-2}\right),\tag{1}$$

where $\overline{\pi}(t)$ is a data. The perturbation will be neglected in the state equation but it has to be retained in the momentum equation.

The next item is to recall the system of equations to be solved. For the sake of simplicity, in this paper we assume that temperature is a given function and hence the energy equation need not to be included. Thus, the compressible Navier–Stokes equations reduce to the mass conservation law, the momentum equation and the equation of state, which are given, respectively, by

$$\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\mathbf{u})=0,\tag{2}$$

$$\frac{\partial\rho\mathbf{u}}{\partial t}+\underbrace{\frac{\partial F_1(\mathbf{u},\rho)}{\partial x}+\frac{\partial F_2(\mathbf{u},\rho)}{\partial y}+\frac{\partial F_3(\mathbf{u},\rho)}{\partial z}}_{\operatorname{div}(\mathbf{F}(\mathbf{u},\rho))}+\nabla\pi-\operatorname{div}\tau=\mathbf{f}(x,y,z,t),\tag{3}$$

$$\overline{\pi}=\rho R\theta.\tag{4}$$

We use standard notations: ρ denotes the density and $\mathbf{u}=(u_1,u_2,u_3)^t$ is the velocity vector. The flux tensor, $\mathbf{F}$, depends on velocity and density; it is given by $F_i(\mathbf{u},\rho)=\rho u_i\mathbf{u}$, $i=1,2,3$. The viscous part of the Cauchy stress tensor is denoted by τ and $\mathbf{f}$ is a generic source term used in the analytical test problems to be considered below. Finally, in the equation of state, $R=\mathcal{R}/\mathcal{M}$ denotes the gas constant, where $\mathcal{R}$ is the universal constant ($\mathcal{R}=8314$ J/(kmol K)), $\mathcal{M}$ is the molecular mass and θ is the absolute temperature which is supposed to be given, may be it has been obtained by solving the energy conservation equation.

From (2) and (4) we get the following divergence condition:

$$\operatorname{div}(\rho\mathbf{u})=-\frac{\partial\rho}{\partial t}=-\frac{\partial}{\partial t}\left(\frac{\overline{\pi}}{R\theta}\right).\tag{5}$$

Please cite this article in press as: A. Bermúdez et al., A projection hybrid finite volume/element method for low-Mach number flows, Journal of Computational Physics (2013), http://dx.doi.org/10.1016/j.jcp.2013.09.029