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Brief paper

Bounds of imaginary spectra of LTI systems in the domain of two of the multiple time delays[☆]

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ABSTRACT

The stability of linear time invariant (LTI) systems with independent multiple time delays and the cluster treatment of characteristic roots (CTCR) paradigm are investigated from a new perspective. It is known that for such systems, all the imaginary characteristic roots can be detected completely on a small set of hypersurfaces in the domain of the delays (Sipahi and Olgac, 2006a). They are called kernel hypersurfaces (KH). The complete description of KH is the only prerequisite for the CTCR stability assessment procedure. As the number of delays increases, however, their evaluation becomes infeasible. Instead, we present a procedure to extract the 2-D cross-sections of these hypersurfaces in the domain of any two of the delays by fixing the remaining delays. In the 2-delay domain of interest, the exact upper and lower bounds of the imaginary spectra are determined. For this, a combination of half-angle tangent representation of the characteristic equation and the Dixon resultant theory is used as the main contributions of this paper. The complete KH are obtained by sweeping the root crossing frequency in this interval. Using this knowledge CTCR creates the cross-section of the stability map in the domain of the two arbitrarily selected delays. We demonstrate the effectiveness of this methodology over an example case study with three independent delays and two commensurate ones.

1. Introduction

This article concerns the asymptotic stability analysis of a general retarded class of linear time-invariant multiple time delay systems (LTI-MTDS):

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \sum_{k=1}^{p} \mathbf{B}_k \mathbf{x}(t - \tau_k) \tag{1}$$

where $\mathbf{x} \in \Re^n$ is the state vector; $\mathbf{A}$, $\mathbf{B}_k$, $k = 1, \ldots, p$ are constant and known matrices in $\Re^{n\times n}$ and p is the number of independent delays in the system. Boldface capital notation is used for vector and matrix quantities. The characteristic equation of this system is

$$g(s, \boldsymbol{\tau}) = \det\left(s\mathbf{I} - \mathbf{A} - \sum_{k=1}^{p} \mathbf{B}_k e^{-\tau_k s}\right) = 0 \tag{2}$$

where $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots, \tau_p) \in \Re^{p+}$ is the delay vector. The delays, at the most stringent case, are assumed rationally independent from each other.

The analysis of asymptotic stability of systems in (2) within the domain of the delays is a fundamental problem in the controls community (Gao, Kammer, Zallugoglu, & Olgac, 2015a,b; Gao, Zalluhoglu, & Olgac, 2014; Gu, Niculescu, & Chen, 2005; Hale, Infante, & Tsen, 1985; Michiels & Niculescu, 2007). It is also known to be an N-P hard class mathematical problem (Toker & Ozbay, 1996). This task becomes much more complex as the number of delays, p, increases (Jarlebring, 2009). For $p \geq 3$, only a limited number of reports exist in the literature (Almodaresi & Bozorg, 2009; Gu & Naghnaeian, 2011; Sipahi & Delice, 2009, 2011; Sipahi, Olgac, & Breda, 2010). First three deal with a similar subclass of MTDS excluding commensurate and cross-talking delay terms in the corresponding characteristic equation. The method in Sipahi and Delice (2011) treats the most general MTDS (1) for determining

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some critical features but fails to declare that this process is only a preparatory phase to an umbrella stability paradigm, and as such it does not stand by itself as a stability method.

In this paper, we deploy the **Cluster Treatment of Characteristic Roots (CTCR)** paradigm for the task (Sipahi & Olgac, 2006). CTCR requires the complete knowledge of the loci in the delay space on which the system exhibits purely imaginary characteristic roots. These loci are composed of two sets: kernel and offspring hypersurfaces (KOH) (Fazelinia, Sipahi, & Olgac, 2007; Sipahi & Olgac, 2006). The kernel hypersurfaces (KH) consist of points that exhibit the smallest positive delay values on all of the p delays. Entire offspring hypersurfaces (OH) are obtained from KH by a point-wise nonlinear transformation (Sipahi & Olgac, 2006) as will be explained later. That is, the mere knowledge of KH is sufficient to obtain the complete infinite set of OH. A number of mathematical procedures are available in the literature to determine the KOH, such as Rekasius substitution (Rekasius, 1980), Kronecker summation (Ergenc, Olgac, & Fazelinia, 2007), and matrix pencil (Niculescu, 1998). For MTDS with $p > 3$. However, the calculation of the KOH in the p-dimensional (p-D) delay space is known to be computationally infeasible (Jarlebring, 2009). Instead, we fix all but two of the delays arbitrarily and examine the intersections of KOH on the domain of the 2 delays. For this, a frequency sweeping technique is deployed, that is completely numerical (Chen & Latchman, 1995). It is recognized to be more effective than those techniques that require symbolic computations especially for systems with higher orders and higher number of delays (Packard & Doyle, 1993). For the frequency sweeping technique, however, precise knowledge of the upper and lower bounds of the imaginary spectra are needed. These bounds are known to exist for the retarded time delay systems (1) (Hale, 1977). The main contribution of this paper is in their determination. To achieve this we deploy a combination of half-angle tangent substitution method (Spivak, 2006) and the Dixon resultant theory (Cayley, 1865; Dixon, 1908; Kapur, Saxena, & Yang, 1994) for the first time in literature.

The paper is structured as follows: Section 2 reviews some key definitions of MTDS and half-angle tangent method. The main results on determining the bounds of the imaginary spectra and the 2-D cross-section of the stability map for MTDS with more than three delays are given in Section 3. Section 4 demonstrates the strength of the proposed methodology over an example case study with three independent and two commensurate delays ($p = 5$).

2. Preliminaries

An LTI-MTDS (1) is asymptotically stable if and only if all its infinitely many characteristic roots are on the left half of the complex plane. The continuity of these roots with respect to delays has already been established in the literature (Hale, 1977; Hale & Lunel, 1993). Since the kernel and offspring hypersurfaces (KOH) are the only loci where the characteristic equation (2) possesses imaginary roots, they are the only potential stability switching locations (thus often referred to as "stability switching hypersurfaces"). For a complete stability map one needs to determine all of these KOH exhaustively. As stated in Section 1, the mere knowledge of KH is sufficient for this. The determination of KH in p-D delay space is, however, computationally infeasible for $p > 3$. Instead, we aim to extract the intersection of the KH on 2-D space of any two of the delays". This would serve a simpler visualization of the problem.

Next, we present some key definitions of time-delayed systems starting with the complete set of the *imaginary* eigenvalues $\mathbf{\Omega}$ of (2) for all possible variations of the delay vector $\boldsymbol{\tau} \in \Re^{p+}$

$$\begin{aligned}\mathbf{\Omega} &= \{\omega | g(s = \omega i, \boldsymbol{\tau}) = 0, \boldsymbol{\tau} \in \Re^{p+}, \omega \in \Re^{+}\} \\ &= \{\omega | \langle \boldsymbol{\tau}, \omega \rangle, \boldsymbol{\tau} \in \Re^{p+}, \omega \in \Re^{+}\} \end{aligned} \tag{3}$$

where $\langle \boldsymbol{\tau}, \omega \rangle$ notation implies that for a specific combination of delays as $\boldsymbol{\tau} \in \Re^{p+}$, there exists an imaginary root, $\omega \in \Re^{+}$ of (2). From the set $\mathbf{\Omega}$ two important definitions arise.

Definition 1 (*Kernel Hypersurfaces (KH)* $\wp_0$). The loci of *all* the points with $\langle \boldsymbol{\tau}, \omega \in \mathbf{\Omega} \rangle$ correspondence and satisfy the constraint $0 < \tau_j \omega < 2\pi$, $j = 1, 2, \ldots, p$ are called the *kernel hypersurfaces*. Notice that the points on these hypersurfaces have the smallest delay compositions which correspond to the same imaginary root ω. ∎

Definition 2 (*Offspring Hypersurfaces (OH)* $\wp$). The hypersurfaces obtained from $\wp_0$ by the following point-wise nonlinear transformation

$$\left\langle \left\{ \tau_1 \pm \frac{2\pi}{\omega} j_1,\ \tau_2 \pm \frac{2\pi}{\omega} j_2, \ldots, \tau_p \pm \frac{2\pi}{\omega} j_p \right\}, \omega \right\rangle,$$
$$j_k = 1, 2, \ldots,\ k = 1, 2, \ldots, p \tag{4}$$

are called the offspring hypersurfaces. The union of KH and OH is defined as KOH, i.e., $\bar{\wp} = \wp_0 \cup \wp$. ∎

Definition 3 (*Root Tendency (RT)*). The root tendency is the transition direction of the imaginary root ωi, when only one of the delays, say τ_j, increases by ε, $0 < \varepsilon \ll 1$, while all the others remain fixed.

$$RT|_{s=\omega i}^{\tau_j} = \operatorname{sgn}\left[\operatorname{Re}\left(\left.\frac{\partial s}{\partial \tau_j}\right|_{s=\omega i}\right)\right]. \tag{5}$$

Clearly $RT = +1$ indicates destabilizing crossing at the imaginary axis and $RT = -1$ implies stabilizing crossing. ∎

Exclusion 1. We limit the analysis in this paper to the most general time-delayed systems for which the imaginary spectra $\mathbf{\Omega}$ entails only simple eigenvalues for all delay compositions $\boldsymbol{\tau} \in \Re^{p+}$. In extremely rare and degenerate cases $\mathbf{\Omega}$ may contain multiple, identical imaginary roots for some particular delay values where the RT in (5) becomes cumbersome to determine. Such degeneracies are kept outside the scope of the paper. If they arise, however, we use a simple deployment of numerical tools (such as QPmR Vyhlidal & Zitek, 2009, DDE-BIFTOOL Engelborghs, Luzyanina, & Roose, 2002) which can reveal the local root transition features near the particular point $\boldsymbol{\tau} \in \Re^{p+}$. Interested readers are also referred to an elaborate treatment on such degeneracies, but for much simpler single delay cases (Chen, Fu, Niculescu, & Guan, 2010a,b). These in-depth investigations also resort to DDE-BIFTOOL to cross-validate their RT determinations.

2.1. Half-angle tangent substitution

The aim is to determine $\mathbf{\Omega}$ on an arbitrarily-selected 2-delay cross-section of KH, take for instance (τ_1, τ_2) without loss of generality, while all the remaining delays $\tau_3, \tau_4, \ldots, \tau_p$ are fixed. For a root $s = \omega i$ we recite the Euler's formula for the transcendental terms in (2)

$$e^{-\tau_k \omega i} = \cos(\upsilon_k) - i \sin(\upsilon_k), \quad \upsilon_k = \tau_k \omega,\ k = 1, 2 \tag{6}$$

and express them in terms of a single parameter, the half-angle tangent:

$$\begin{aligned}&\cos(\upsilon_k) = \frac{1 - z_k^2}{1 + z_k^2}, \qquad \sin(\upsilon_k) = \frac{2 z_k}{1 + z_k^2}, \\ &z_k = \tan\left(\frac{\upsilon_k}{2}\right), \quad k = 1, 2.\end{aligned} \tag{7}$$