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Brief paper

New stability conditions for a class of linear time-varying systems[☆]

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ABSTRACT

This paper considers the problem of asymptotic stability for linear time-varying systems of the form $\dot{x}(t) = A(t)x(t)$. Some new stability conditions are proposed. First, two stability conditions for nonlinear time-varying systems are given by using non-monotonic Lyapunov functions. Then the results obtained are extended to the linear case, two stability conditions with infinite integral are derived. Furthermore, by using the top-floor function, a linear matrix inequalities (LMI) condition and an eigenvalue criterion for asymptotic stability of systems are presented. Comparing with the existing results, the conditions obtained allow both $A(t)$ and $\dot{A}(t)$ are unbounded at $t = +\infty$. Some numerical examples are provided to show the effectiveness of the theoretical results.

1. Introduction

The stability of linear time-varying (LTV) systems has been an active research field in science and engineering for many years. While important, LTV systems are very hard to investigate despite the fundamental stability problem. It is well known that, a linear time invariant system is stable if the eigenvalues have negative real part. However, for linear time-varying systems, even though all the pointwise eigenvalues have negative real parts (Rosenbrock, 1963; Rugh, 1993), the time-varying system still may be unstable. Thus the stability problem of LTV systems is much more complicated than that of linear time invariant systems.

Numerous important results, including but not limited to Amato, Celentano, and Garofalo (1993), Desoer (1969), Garcia, Peres, and Tarbouriech (2010), Guo and Rugh (1995), Ilchmann, Owens, and Prätzel-Wolters (1987), Jetto and Orsini (2009), Kamen, Khargonekar, and Tannembaum (1989), Krause and Kumar (1986), Mullhaupt, Buccieri, and Bonvin (2007), Solo (1994) and Tan and Duan (2009), have been obtained through the efforts of researchers. It is worth noting that most of existing stability assessment methods (Amato et al., 1993; Desoer, 1969; Guo & Rugh, 1995; Ilchmann et al., 1987; Kamen et al., 1989; Krause & Kumar, 1986; Mullhaupt et al., 2007; Rosenbrock, 1963; Solo, 1994) tend to take the pointwise stability as a necessary precondition, or impose some "slowly varying" conditions to the system, i.e., impose a bound on $\|\dot{A}(t)\|$. In addition, most of existing results (Amato et al., 1993; Desoer, 1969; Garcia et al., 2010; Guo & Rugh, 1995; Ilchmann et al., 1987; Jetto & Orsini, 2009; Kamen et al., 1989; Krause & Kumar, 1986; Mullhaupt et al., 2007; Rosenbrock, 1963; Solo, 1994; Tan & Duan, 2009) also require $A(t)$ to be bounded, i.e., $\exists\, \xi > 0$ such that $\|A(t)\| \le \xi$ for $t \ge 0$.

In this paper, we will remove these restrictions to study the stability for a class of LTV systems, where both $A(t)$ and $\dot{A}(t)$ may be unbounded at $t = +\infty$. The main contributions of this paper include some new stability conditions: (i) two stability conditions for nonlinear time-varying systems are given by using non-monotonic Lyapunov function, (ii) two stability conditions with infinite integral for LTV systems are derived, and (iii) a LMI condition and an eigenvalue criterion for asymptotic stability of LTV systems are presented by using the top-floor function.

2. Preliminaries

Let R denote the set of real numbers, $R^+ = [0, +\infty)$, R^n be the n-dimensional real space, $C[R^+, R^+]$ be the set of continuous maps from R^+ to R^+. A function $w \in C[R^+, R^+]$ is said to belong to class K, if $w(0) = 0$ and w is strictly increasing. $\|x\|$ denotes the Euclidean norm of x. $\|A\|$ denotes the induced norm of matrix A. For an $n \times n$ real symmetric matrix A, $\lambda_i(A)$, $1 \le i \le n$, denotes the ith eigenvalue of matrix A. Specially, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ represent the maximum and minimum eigenvalues of matrix A, respectively.

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3. Stability conditions for nonlinear time-varying systems

Consider the following nonlinear time-varying system:

$$\dot{x}(t) = f(t, x(t)), \quad t \geq t_0 \tag{1}$$

where $t_0 \geq 0$ and initial condition $x_0 = x(t_0)$, where $x(t) \in R^n$ is the state of systems, $f : R^+ \times R^n \to R^n$ is a continuous function satisfying $f(t, 0) = 0$ for all $t \geq t_0$.

Definition 1 (*Khalil, 2002*)**.** The system (1) is said to be:

- *Lyapunov stable* if for each $\varepsilon > 0$, there is $\delta(\varepsilon, t_0) > 0$ such that

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall t \geq t_0 \geq 0. \tag{2}$$

- *uniformly stable* if for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$, independent of t_0, such that (2) is satisfied.
- *asymptotically stable* if it is *Lyapunov stable* and there is a constant $c = c(t_0) > 0$ such that $x(t) \to 0$ as $t \to \infty$, for all $\|x(t_0)\| < c$.
- *uniformly asymptotically stable* if it is *uniformly stable* and there is a constant $c > 0$, independent of t_0, such that for all $\|x(t_0)\| < c$, $x(t) \to 0$ as $t \to \infty$, uniformly in t_0; that is, for each $\eta > 0$, there is $T = T(\eta) > 0$ such that

$$\|x(t)\| < \eta, \quad \forall t \geq t_0 + T(\eta), \ \forall \|x(t_0)\| < c. \tag{3}$$

- *exponentially stable* if there exist positive constants c, k and a such that

$$\|x(t)\| \leq k \|x(t_0)\| e^{-a(t-t_0)}, \quad \forall \|x(t_0)\| < c. \tag{4}$$

Lemma 1. *Consider the system (1). Suppose there exist a continuously differentiable function $V : R^+ \times R^n \to R^+$ with $V(t, 0) = 0$ for $t \geq t_0$, a class-K function α, a function $w \in C[R^+, R^+]$ with $w(0) = 0$, a constant $\eta > 0$, a positive definite function r, and a continuous function $g : R^+ \to R$ such that*

(1) $\alpha(\|x(t)\|) \leq V(t, x)$,
(2) $V(t, x) \leq w(V(t_0, x(t_0)))$,
(3) $\dot{V}(t, x) \leq g(t) r(V(t, x))$ *with* $\int_{t_0}^{+\infty} g(t)\, dt = -\infty$,

for any $\|x(t_0)\| < \eta$. Then the system (1) is asymptotically stable.

Proof. *Lyapunov stability*: For any given $\varepsilon > 0$, it follows from the definitions of V and w that there exists a sufficiently small $0 < \delta(\varepsilon, t_0) < \eta$ such that

$$w(V(t_0, x(t_0))) \leq \alpha(\varepsilon)$$

for all $\|x_0\| \leq \delta$. Combining conditions (1) and (2), we have

$$\alpha(\|x(t)\|) \leq V(t, x) \leq w(V(t_0, x(t_0))) \leq \alpha(\varepsilon)$$

for $t \geq t_0$, i.e, $\|x(t)\| \leq \varepsilon$, $t \geq t_0$, and hence the system (1) is Lyapunov stable.

Asymptotic convergence: By using the contradiction and condition (3), we have that the system (1) is asymptotically stable. This completes the proof.

Lemma 2. *Consider the system (1). Suppose there exist a continuously differentiable function $V : R^+ \times R^n \to R^+$ with $V(t, 0) = 0$ for $t \geq t_0$, a class-K function α, and a continuous function $g : R^+ \to R$ such that*

(1) $\alpha(\|x(t)\|) \leq V(t, x)$,
(2) $\dot{V}(t, x) \leq g(t) V(t, x)$ *with* $\int_{t_0}^{+\infty} g(t)\, dt = -\infty$.

then the system (1) is asymptotically stable.

Proof. *Lyapunov stability*: It follows from condition (2) that

$$\frac{\dot{V}(t, x)}{V(t, x)} \leq g(t). \tag{5}$$

By integrating both sides of (5) from t_0 to t, we obtain

$$\int_{t_0}^{t} \frac{\dot{V}(s, x(s))}{V(s, x(s))}\, ds \leq \int_{t_0}^{t} g(s)\, ds. \tag{6}$$

That is,

$$V(t, x) \leq V_0 e^{\int_{t_0}^{t} g(s)\, ds}. \tag{7}$$

Let $G(t) = \int_{t_0}^{t} g(s)\, ds$, then it follows from the definition of g and condition (2) that $G(t)$ is continuous and $G(t) \to -\infty$ as $t \to +\infty$, which means that for a given $a > 0$, there exists a $T \geq t_0$ such that $G(t) \leq -a$ for $t \geq T$.

Let $A = \max_{t_0 \leq t \leq T}\{G(t)\}$ and $B = \max\{-a, A\}$, then it follows from (7) that

$$V(t, x) \leq V_0 e^{\int_{t_0}^{t} g(s) ds} = V_0 e^{G(t)} \leq V_0 e^{B} \tag{8}$$

for $t \geq t_0$. Now for any given $\varepsilon > 0$, it follows from the definitions of V and B that there exists a $\delta(\varepsilon) > 0$ such that

$$V_0 e^{B} \leq \alpha(\varepsilon) \tag{9}$$

for $\|x_0\| \leq \delta$. Combining (8), (9) and condition (1), we have

$$\alpha(\|x(t)\|) \leq V(t, x) \leq V_0 e^{B} \leq \alpha(\varepsilon) \tag{10}$$

which implies that $\|x(t)\| \leq \varepsilon$, $t \geq t_0$.

Asymptotic convergence: It follows from condition (2) and (7) that $V(t, x) \to 0$ as $t \to +\infty$, i.e., the system (1) is asymptotically stable. This completes the proof.

Remark 1. The conventional Lyapunov stability theory requires the Lyapunov function $V(t, x)$ of systems is monotonic decreasing to ensure the asymptotic stability of systems. However, Lemma 2 (or Lemma 1) only needs the derivative of $V(t, x)$ satisfies the condition (2) (or condition (3)), which allows $V(t, x)$ to be increasing on some time intervals. Therefore the conditions of Lemma 2 (or Lemma 1) are less conservative than that of the conventional Lyapunov stability theory.

Example 1. Consider the nonlinear time-varying system given by

$$\dot{x}(t) = 0.1(\sin t - 0.8) x^3(t), \quad t \geq t_0 \tag{11}$$

where $t_0 \geq 0$ and $0 \leq x_0 \leq 1$. Let $V(t, x) = x^2(t)$, $\alpha(s) = \frac{1}{2}s^2$, $w(s) = 2s$, $g(t) = 0.2(\sin t - 0.8)$ and $r(s) = s^2$, then we have

$$\dot{V}(t, x) = 0.2(\sin t - 0.8) x^4(t) = g(t) \cdot r(V(t, x)) \tag{12}$$

and $\int_{t_0}^{+\infty} g(t)\, dt = -\infty$, which means that the system (11) is asymptotically stable from Lemma 1. Choosing $t_0 = 0$ and $x_0 = 1$, the state $x(t)$ of systems versus time is shown in Fig. 1.

In fact, the explicit solution of (11) is:

$$x(t) = \frac{x_0}{\sqrt{1 + 0.2x_0^2(\cos t - \cos t_0 + 0.8(t - t_0) - 1)}}$$

for $t \geq t_0$.

Remark 2. The stability of the system (11) cannot be easily studied by the conventional Lyapunov stability theory since it is difficult to construct a monotonic decreasing Lyapunov function for the system (11).