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New finite-sum inequalities with applications to stability of discrete time-delay systems[☆]

Le Van Hien[a,b], Hieu Trinh[a]

[a] School of Engineering, Deakin University, Geelong, VIC 3217, Australia

[b] Department of Mathematics, Hanoi National University of Education, Hanoi, Viet Nam

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ABSTRACT

This paper is concerned with the problem of stability analysis of discrete time-delay systems. New finite-sum inequalities, which encompass the ones based on Abel lemma or Wirtinger type inequality, are first proposed. The potential capability of the newly derived inequalities is then demonstrated by establishing less conservative stability conditions for some classes of linear discrete-time systems with delay. The derived stability criteria are theoretically and numerically proved to be less conservative than existing results.

1. Introduction

Consider a linear discrete time-delay system of the form

$$\begin{cases} x(k+1) = Ax(k) + A_d x(k-d(k)), \quad k \geq 0, \\ x(k) = \phi(k), \quad k = -d_2, -d_2+1, \dots, 0, \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the system state, $A, A_d \in \mathbb{R}^{n\times n}$ are given matrices, $d(k)$ is a time-varying delay satisfying $d_1 \leq d(k) \leq d_2$, where d_1, d_2 are known positive integers and $\phi(k)$ is an initial condition.

System (1) frequently appears in engineering because of many practical control systems are implemented through a network in which communication delays occur in the control channel (Huang & Nguang, 2009; Shu & Lin, 2014). The existence of time-delay usually is a source of oscillations, poor performance or instability. Therefore, during the last decade, the problem of stability analysis and applications to control of system (1) has received considerable attention. To mention a few, we refer the reader to Feng, Lam, and Yang (2015), Hien, An, and Trinh (2014), Kim (2015), Kwon, Park, Park, Lee, and Cha (2013), Meng, Lam, Du, and Gao (2010), Nam, Pubudu, and Trinh (2015), Shao and Han (2011) and Zhang and Han (2015).

For the case of a constant delay, analytical method based on the characteristic equation or a lifting technique can be used to derive a necessary and sufficient stability condition for system (1). However, in many practical systems, time-delay is usually random but bounded in a certain range (Shu & Lin, 2014), and thus, the analytical method or lifting technique is no longer suitable. An alternative effective approach for stability analysis of system (1) is the use of the Lyapunov–Krasovskii functional (LKF) method (see, for example, Hien et al., 2014; Kim, 2015; Zhang & Han, 2015). Based on a priori construction of an LKF combining with some bounding techniques, sufficient conditions are derived in terms of linear matrix inequalities (LMIs) ensuring asymptotic stability of system (1). For example, a widely used LKF candidate is constructed as

$$V_0(k) = d\sum_{s=-d}^{-1}\sum_{j=k+s}^{k} u^T(j)Ru(j),$$

where d is a positive integer, k is an integer representing time variable, R is a positive definite matrix and $u(k)$ is the difference at time k of the state $x(k)$ defined as $u(k) = \Delta x(k) \triangleq x(k+1) - x(k)$. The difference of $V_0(k)$ is given by

$$\Delta V_0(k) = d^2 u^T(k)Ru(k) - d\sum_{s=k-d}^{k-1} u^T(s)Ru(s). \tag{2}$$

To derive LMI-based stability conditions from (2), it is required to find a lower bound of the summation term $d\sum_{s=k-d}^{k-1} u^T(s)Ru(s)$.

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E-mail addresses: hienlv@hnue.edu.vn (L.V. Hien), hieu.trinh@deakin.edu.au (H. Trinh).

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More generally, for integers $a < b$, a function $u : \mathbb{Z}[a, b] \to \mathbb{R}^n$ and a positive definite matrix R, a lower bound of a finite-sum in the form

$$S_R^u(a, b) = \sum_{k=a}^{b} u^T(k)Ru(k) \tag{3}$$

plays an important role in establishing stability conditions for system (1). In addition, a tighter lower bound of (3) definitely can be helpful to derive less conservative stability conditions. By using the Jensen-type inequality, the term $S_R^u(a, b)$ in (3) is estimated as follows

$$S_R^u(a, b) \ge \frac{1}{\ell}\left(\sum_{k=a}^{b} u(k)\right)^T R\left(\sum_{k=a}^{b} u(k)\right), \tag{4}$$

where $\ell = b-a+1$ denotes the length of interval $[a, b]$ in $\mathbb{Z}$. Similar to (4) we can obtain the following double summation inequality

$$\sum_{k=a}^{b}\sum_{s=a}^{k} u^T(s)Ru(s) \ge \frac{2}{\ell(\ell+1)}\left(\sum_{k=a}^{b}\sum_{s=a}^{k} u(s)\right)^T R\left(\sum_{k=a}^{b}\sum_{s=a}^{k} u(s)\right). \tag{5}$$

An interesting improvement of (5) has recently been achieved by utilizing Abel lemma (Zhang & Han, 2015) or Wirtinger-type inequality (Nam et al., 2015; Seuret, Gouaisbaut, & Fridman, 2015) which is given by

$$S_R^u(a, b) \ge \frac{1}{\ell}\upsilon_1^T R\upsilon_1 + \frac{3(\ell+1)}{\ell(\ell-1)}\left(\upsilon_1 - \frac{2}{\ell+1}\upsilon_2\right)^T R\left(\upsilon_1 - \frac{2}{\ell+1}\upsilon_2\right), \tag{6}$$

where $\upsilon_1 = \sum_{k=a}^{b} u(k)$ and $\upsilon_2 = \sum_{k=a}^{b}\sum_{s=a}^{k} u(s)$.

Clearly, (6) gives a tighter bound for (3) than (4) does. Moreover, as discussed in Zhang and Han (2015), how to find a new lower bound for (3) is of significance in theory and practice, which motivates our present study.

In this paper, we first propose some new finite-sum inequalities in single and double forms. The obtained inequality in single form theoretically encompasses the existing one given in (6). The proposed inequalities are then employed to derive delay-dependent stability conditions for system (1). Finally, two examples are provided to show the effectiveness and significant improvement of our results over the existing literature.

2. New finite-sum inequalities

Hereafter, for given integers $a < b$ and a function $u : \mathbb{Z}[a, b] \to \mathbb{R}^n$, we denote $\ell = b - a + 1$, $\upsilon_1 = \sum_{k=a}^{b} u(k)$, $\upsilon_2 = \sum_{k=a}^{b}\sum_{s=a}^{k} u(s)$ and $\upsilon_3 = \sum_{k=a}^{b}\sum_{s=a}^{k}\sum_{i=a}^{s} u(i)$. We also denote $\mathbb{S}_+^n$ the set of symmetric positive definite matrices in $\mathbb{R}^{n\times n}$.

Lemma 1. *For a matrix $R \in \mathbb{S}_n^+$, integers $a < b$ and a function $u : \mathbb{Z}[a, b] \to \mathbb{R}^n$, the following inequality holds*

$$S_R^u(a, b) \ge \frac{1}{\ell}\upsilon_1^T R\upsilon_1 + \frac{3(\ell+1)}{\ell(\ell-1)}\zeta_1^T R\zeta_1 + \frac{5(\ell+1)(\ell+2)^2}{\ell(\ell-1)(\ell^2+11)}\zeta_2^T R\zeta_2, \tag{7}$$

where $\zeta_1 = \upsilon_1 - \frac{2}{\ell+1}\upsilon_2$ and $\zeta_2 = \upsilon_1 - \frac{6}{\ell+1}\upsilon_2 + \frac{12}{(\ell+1)(\ell+2)}\upsilon_3$.

Proof. Inspired by Hien and Trinh (2015), we define an approximation function $v : \mathbb{Z}[a, b] \to \mathbb{R}^n$ by

$$v(k) = u(k) - \frac{1}{\ell}\upsilon_1 + \alpha(k)\chi_1 + \beta(k)\chi_2, \tag{8}$$

where $\alpha(k) = (k-a) + \frac{1-\ell}{2}$, $\beta(k) = (k-a)^2 - \ell(k-a) + \frac{\ell^2-1}{6}$, $k \in \mathbb{Z}[a, b]$, and $\chi_1, \chi_2 \in \mathbb{R}^n$ are constant vectors which will be defined later. Since $\sum_{k=a}^{b}\alpha(k) = 0$, $\sum_{k=a}^{b}\beta(k) = 0$, it follows from (8) that

$$\begin{aligned}\sum_{k=a}^{b} v^T(k)Rv(k) &= J(u) + 2\chi_1^T R\left(\sum_{k=a}^{b}\alpha(k)u(k)\right) \\ &\quad + 2\chi_2^T R\left(\sum_{k=a}^{b}\beta(k)u(k)\right) + \left(\sum_{k=a}^{b}\alpha^2(k)\right)\chi_1^T R\chi_1 \\ &\quad + \left(\sum_{k=a}^{b}\beta^2(k)\right)\chi_2^T R\chi_2 + 2\left(\sum_{k=a}^{b}\alpha(k)\beta(k)\right)\chi_1^T R\chi_2,\end{aligned} \tag{9}$$

where $J(u) = S_R^u(a, b) - \frac{1}{\ell}\upsilon_1^T R\upsilon_1$.

Let $\hat{u}(k) = \sum_{i=a}^{k-1} u(i)$ for $k > a$ and $\hat{u}(a) = 0$ then $\alpha(k)u(k) = \Delta(\alpha(k)\hat{u}(k)) - \hat{u}(k+1)$. Therefore

$$\sum_{k=a}^{b}\alpha(k)u(k) = \frac{\ell+1}{2}\upsilon_1 - \sum_{k=a}^{b}\hat{u}(k+1) = \frac{\ell+1}{2}\zeta_1. \tag{10}$$

Similar to (10) we have

$$\sum_{k=a}^{b}\beta(k)u(k) = \frac{\ell^2-1}{6}\upsilon_1 - \sum_{k=a}^{b}\hat{u}(k+1)\Delta\beta(k). \tag{11}$$

Let $\tilde{u}(k) = \sum_{s=a}^{k}\hat{u}(s)$ then $\hat{u}(k+1)\Delta\beta(k) = \Delta(\Delta\beta(k)\tilde{u}(k)) - 2\tilde{u}(k+1)$. Note also that $\sum_{k=a}^{b}\tilde{u}(k+1) = \upsilon_3$, (11) leads to

$$\sum_{k=a}^{b}\beta(k)u(k) = \frac{1}{6}\xi, \tag{12}$$

where $\xi = (\ell^2-1)\upsilon_1 - 6(\ell+1)\upsilon_2 + 12\upsilon_3$.

On the other hand, a direct computation gives $\sum_{k=a}^{b}\alpha^2(k) = \frac{\ell(\ell^2-1)}{12}$, $\sum_{k=a}^{b}\alpha(k)\beta(k) = -\frac{\ell(\ell^2-1)}{12}$ and $\sum_{k=a}^{b}\beta^2(k) = \frac{\ell(\ell^2-1)(\ell^2+11)}{180}$. By injecting those equalities into (9), combining with (10), (12) and the fact that $\sum_{k=a}^{b} v^T(k)Rv(k) \ge 0$, we obtain

$$J(u) + (\ell+1)\chi_1^T R\zeta_1 + \frac{\ell(\ell^2-1)}{12}\chi_1^T R\chi_1 + \mathcal{R} \ge 0, \tag{13}$$

where $\mathcal{R} = \frac{1}{3}\chi_2^T R\xi - \frac{\ell(\ell^2-1)}{6}\chi_1^T R\chi_2 + \frac{\ell(\ell^2-1)(\ell^2+11)}{180}\chi_2^T R\chi_2$.

Now, we define $\chi_1 = -\lambda\zeta_1$, where λ is a real scalar, it follows from (13) that

$$J(u) \ge (\ell+1)\left[\lambda - \frac{\ell(\ell-1)}{12}\lambda^2\right]\zeta_1^T R\zeta_1 - \mathcal{R}. \tag{14}$$

The function $\lambda - \frac{\ell(\ell-1)}{12}\lambda^2$ attains its maximum $\frac{3}{\ell(\ell-1)}$ at $\lambda = \frac{6}{\ell(\ell-1)}$. Then, by (14), $J(u) \ge \frac{3(\ell+1)}{\ell(\ell-1)}\zeta_1^T R\zeta_1 - \mathcal{R}$. In addition, by injecting $\chi_1 = \frac{-6}{\ell(\ell-1)}\zeta_1$ into $\mathcal{R}$ we then obtain

$$J(u) \ge \frac{3(\ell+1)}{\ell(\ell-1)}\zeta_1^T R\zeta_1 - \frac{\ell(\ell^2-1)(\ell^2+11)}{180}\chi_2^T R\chi_2 - \frac{(\ell+1)(\ell+2)}{3}\chi_2^T R\zeta_2. \tag{15}$$