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Consensus of generalized integrators: Convergence rate and disturbance attenuation property[☆]

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ABSTRACT

We study a generalized integrator consensus network, each node of which is a single-integrator cascaded with some heterogeneous non-integrator internal dynamics. Under the assumption that the interconnection graph is undirected and connected, we first investigate the convergence property of the consensus network and show that the interconnection is beneficial for the enhancement of the convergence rate if the non-integrator internal dynamics of each node is strictly passive. We then show that the interconnection is also advantageous for the disturbance attenuation in the consensus network.

1. Introduction

A significant amount of research efforts have been focused on consensus of multi-agent systems. One performance index for a consensus network is its convergence rate, which characterizes the property of how fast its individual systems reach an agreement on their variables of interest. Olfati-Saber and Murray (2004) have showed that the convergence rate in a single-integrator network with an undirected topology can be characterized by the second smallest eigenvalue of the graph Laplacian matrix associated with the network topology, which is known as the algebraic connectivity. For a general network, when the individual systems are identical and linear time-invariant, one can decompose the network dynamics into low-order systems as done in Fax and Murray (2002) to investigate the convergence rate of the network.

Another key performance index for a consensus network is its disturbance attenuation property. When each node is subject to disturbance, it is hardly expected to achieve perfect consensus. In such a case, it is important to ensure some network performance relative to the perfect consensus. Local controller design for the enhancement of the disturbance attenuation property has been studied in Li, Duan, and Chen (2011); Liu, Jia, Du, and Yuan (2009) for consensus networks consisting of identical linear systems. Oh, Moore, and Ahn (2014) have addressed the network interconnection design and the local controller design for enhancing the disturbance attenuation property of a consensus network of linear systems.

The majority of the works described above have been focused on the performance of consensus networks consisting of identical systems. Though consensus of heterogeneous systems has been recently studied in Kim, Shim, and Seo (2011); Lunze (2012); Wieland, Sepulchre, and Allgöwer (2011), the convergence rate and disturbance attenuation property have yet to be studied. Based on this observation, we attempt to investigate the convergence rate and disturbance attenuation in a generalized integrator network, each node of which is a single-integrator cascaded with additional heterogeneous internal dynamics and subject to exogenous disturbance as showed in Fig. 1. In the remainder of this paper,

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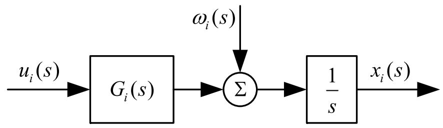

Fig. 1. Block diagram for generalized integrator node.

we will refer to these additional heterogeneous dynamics as the *non-integrator internal dynamics*. Here the control input for each node is based on relative variables through undirected network topology as clarified below. Though the generalized integrator network has been studied in Moore, Vincent, Lashhab, and Liu (2011); Oh, Lashhab, Moore, Vincent, and Ahn (2015); Wang and Elia (2010), only consensus conditions have been investigated in those works without consideration of the convergence rate and disturbance attenuation properties.

Accordingly, contributions of this work can be summarized as follows: Based on the Lyapunov analysis, we show that it is possible to analytically find the worst performance bound for the convergence rate and disturbance attenuation of the generalized integrator network if the interconnection graph is undirected and the non-integrator internal dynamics of each node is strictly passive. Further we show that one can enhance the performance bound by increasing the second smallest eigenvalue of the Laplacian matrix, which can be utilized for network design.

Notations: The following notations are used throughout the paper. The set of real numbers is denoted by $\mathbb{R}$. We denote $[1 \cdots 1]^T \in \mathbb{R}^n$ by 1_n. The $n \times n$ identity matrix is denoted by I_n. For two real vectors x and ξ, (x, ξ) denotes the stacked column vector $[x^T \;\; \xi^T]^T$. For a real square matrix A, we denote the positive definiteness (respectively, positive semi-definiteness) of A by $A \succ 0$ (respectively, $A \succeq 0$). For a real matrix A, A^T denotes the transpose of A. For a square matrix A, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the smallest and the largest eigenvalues of the matrix, respectively. Further, $\lambda_i(A)$ denotes the ith smallest eigenvalue.

An undirected weighted graph $\mathcal{G}$ is defined as a triple $\mathcal{G} := (\mathcal{V}, \mathcal{E}, \mathcal{W})$, where $\mathcal{V}$ denotes the set of nodes, $\mathcal{E} \subseteq \{\{i, j\} : i, j \in \mathcal{V}\}$ denotes the set of edges, and $\mathcal{W} : \mathcal{E} \to \bar{\mathbb{R}}_+$ denotes a map assigning nonnegative real numbers to the edges. The nonnegative value $\mathcal{W}(\{i, j\})$ assigned to $\{i, j\} \in \mathcal{E}$ is called the weight of the edge. We assume that there are no self-loops, i.e., for any $i \in \mathcal{V}$, $\{i, i\} \notin \mathcal{E}$. If $\{i, j\} \in \mathcal{E}$, j (respectively, i) is said to be a neighbor of i (respectively, j). The set of neighbors of $i \in \mathcal{V}$ is defined as $\mathcal{N}_i := \{j \in \mathcal{V} : \{i, j\} \in \mathcal{E}\}$. A path between two nodes is a sequence of edges by which it is possible to move along the sequence of the edges from one of the nodes to the other node. If there exists at least one path from any node to any other nodes in $\mathcal{G}$, the graph is said to be connected. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ be an undirected weighted graph with N nodes. The Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{N \times N}$ of $\mathcal{G}$ is defined as

$$l_{ij} := \begin{cases} \sum_{j \in \mathcal{N}_i} w_{ij}, & i = j, \\ -w_{ij}, & \{i, j\} \in \mathcal{E}, \\ 0, & \{i, j\} \notin \mathcal{E}, \end{cases}$$

where $w_{ij} := \mathcal{W}(\{i, j\})$ for any $\{i, j\} \in \mathcal{E}$.

2. Generalized single-integrator network

Consensus in a network of single-integrator nodes has been actively studied in the literature (Jadbabaie, Lin, & Morse, 2003; Lin, Francis, & Maggiore, 2007; Moreau, 2005; Olfati-Saber & Murray, 2004; Ren, Beard, & McLain, 2005). In those works, the following network over a graph $\mathcal{G}$ has been considered:

$$\dot{x}_i = u_i, \quad u_i = \sum_{j \in \mathcal{N}_i} w_{ij}(x_j - x_i), \; i = 1, \ldots, N, \tag{1}$$

where $x_i \in \mathbb{R}$ is the output, $u_i \in \mathbb{R}$ is the input, $\mathcal{N}_i$ is the set of neighbors of node i, and w_{ij} is the weight for the corresponding edge. We focus on networks with undirected interconnection topology in this paper. We say that the single-integrator network (1) asymptotically reaches consensus if $x_i(t) - x_j(t) \to 0$ as $t \to \infty$ for all $i = 1, \ldots, N$.

From the network (1), we construct the following generalized integrator consensus network by considering additional non-integrator internal dynamics contained in the individual nodes and exogenous disturbances injected into the nodes:

$$\dot{x}_i = \zeta_i + \omega_i, \tag{2a}$$

$$\dot{\xi}_i = A_i \xi_i + B_i u_i, \tag{2b}$$

$$\zeta_i = C_i \xi_i + D_i u_i, \tag{2c}$$

$$u_i = \sum_{j \in \mathcal{N}_i} w_{ij}(x_j - x_i), \tag{2d}$$

where $x_i \in \mathbb{R}$, $\xi_i \in \mathbb{R}^{n_i}$, $\zeta_i \in \mathbb{R}$, and $\omega_i \in \mathbb{R}$ for each $i = 1, \ldots, N$. Note that the order of the non-integrator internal dynamics of node i is not necessarily identical.

As discussed in Oh et al. (2015), the generalized integrator network (2) captures characteristics of some physical networks. An example is the load frequency control (LFC) network of an electrical power grid. In the LFC network, the output of each individual system is the phase of its voltage, which is the integration of the angular velocity. The interconnection is power exchanges among the individual systems through transmission lines, which are dependent on phase differences. Further individual systems have non-integrator internal dynamics related to their governor, turbine, generator, and local controller.

Consensus conditions for the network (2) has been studied in Oh et al. (2015), which have showed that the network asymptotically reaches consensus under the disturbance free condition, i.e., $\omega_i \equiv 0$ for all $i = 1, \ldots, N$, if $\mathcal{G}$ is connected and the $G_i(s) = C_i(sI_{n_i} - A_i)^{-1}B_i + D_i$ is weakly strictly positive real (Brogliato, Lozano, Maschke, & Egeland, 2007). However the convergence rate for the network (2) has yet to be studied. Further the effect of exogenous disturbance needs to be studied because most of physical systems are subject to some disturbance.

3. Main result

3.1. Convergence rate

To investigate convergence rate of the network (2), we introduce the following assumption:

Assumption 3.1. For the network (2), assume the following:

- The graph $\mathcal{G}$ is undirected and connected.
- For $i = 1, \ldots, N$, there exist $m_i > 0$, $P_i \in \mathbb{R}^{n_i \times n_i}$, $P_i = P_i^T \succ 0$, $R_i \in \mathbb{R}^{n_i \times 1}$, and $W_i \in \mathbb{R}$ such that

$$P_i A_i + A_i^T P_i = -R_i R_i^T - m_i P_i, \tag{3a}$$

$$P_i B_i - C_i^T = -R_i W_i, \tag{3b}$$

$$D_i + D_i^T = W_i^T W_i. \tag{3c}$$

- For $i = 1, \ldots, N$, $D_i \neq 0$.

The second condition in Assumption 3.1 means that the non-integrator internal dynamics consisting of (2b) and (2c) satisfies the strict passivity condition (Khalil, 2002). It follows from the Kalman–Yakubovich–Popov lemma that $G_i(s) = C_i(sI_{n_i} - A_i)B_i + D_i$ is strictly positive real. Thus, under Assumption 3.1, the network (2) asymptotically reaches consensus (Oh et al., 2015).

Let $x = [x_1 \cdots x_N]^T$, $\xi = [\xi_1^T \cdots \xi_N^T]^T$, $\zeta = [\zeta_1 \cdots \zeta_N]^T$, and $\omega = [\omega_1 \cdots \omega_N]^T$. Further let $A = \text{blkdiag}(A_1, \ldots, A_N)$, $B = \text{blkdiag}$