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Brief paper

Control of the Landau–Lifshitz equation☆

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ABSTRACT

The Landau–Lifshitz equation describes the dynamics of magnetization inside a ferromagnet. This equation is nonlinear and has an infinite number of stable equilibria. It is desirable to control the system from one equilibrium to another. A control that moves the system from an arbitrary initial state, including an equilibrium point, to a specified equilibrium is presented. It is proven that the second point is an asymptotically stable equilibrium of the controlled system. The results are illustrated with some simulations.

1. Introduction

The Landau–Lifshitz equation is a partial differential equation (PDE), which describes the magnetic behaviour within ferromagnetic structures. This equation was originally developed to model the behaviour of domain walls, which separate magnetic regions within a ferromagnet (Landau & Lifshitz, 2008). Ferromagnets are often found in memory storage devices such as hard disks, credit cards or tape recordings. Each set of data stored in a memory device is uniquely assigned to a specific stable magnetic state of the ferromagnet, and hence it is desirable to control magnetization between different stable equilibria. This is difficult due to the presence of hysteresis in the Landau–Lifshitz equation. Hysteresis indicates the presence of multiple equilibria (Chow & Morris, 2014; Morris, 2011). Because of this, a particular control can lead to different magnetizations; that is, the particular path of magnetization depends on the initial state of the system and looping in the input–output map is typical (Chow & Morris, 2014; Morris, 2011).

There is now an extensive body of results on control and stabilization of linear PDE's; see for instance the books (Bensoussan, 2007; Curtain & Zwart, 1995; Lasiecka & Triggiani, 2000a,b) and the review paper (Morris, 2010). Stability results for the Landau–Lifshitz equation are often based on linearization (Carbou & Labbé, 2006a,b; Jizzini, 2011; Labbe, Privat, & Trelat, 2012). In these works, the spectral properties of the linear operator are determined. In Mayergoyz, Serpico, and Bertotti (2010), sufficient assumptions are made that simplify a general form of the Landau–Lifshitz equation into an ordinary differential equation; and based on this, the magnetization dynamics are shown to be stable.

The magnetic state of a ferromagnet can be changed by an applied magnetic field, which is viewed as the control. From this physically meaningful perspective, the control enters the Landau–Lifshitz equation nonlinearly. In Carbou and Labbe (2012), the Landau–Lifshitz equation is linearized and shown to have an unstable equilibrium; and to stabilize this equilibrium a control that is the average of the magnetization in one direction and zero in the other two directions is used. In Carbou, Labbé, and Trélat (2008, 2009), solutions to the Landau–Lifshitz equation are shown to be arbitrarily close to domain walls given a constant control. Experiments and numerical simulations demonstrating the control of domains walls in a nanowire are presented in Noh, Miyamoto, Okuda, Hayashi, and Kim (2012) and Wieser, Vedmedenko, and Wiesendanger (2011).

In the next section, the uncontrolled Landau–Lifshitz equation is described. It is known to have multiple stable equilibria (Guo & Ding, 2008, Theorem 6.1.1). In Section 3, a control, acting as the applied magnetic field, is introduced into the Landau–Lifshitz equation nonlinearly. The control objective is to steer the system dynamics between stable equilibrium points. Results demonstrate the controlled Landau–Lifshitz equation is stable, and the linearized controlled Landau–Lifshitz equation is asymptotically stable. In Section 4, simulations for the full equation are presented.

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2. Landau–Lifshitz equation

Consider the magnetization

$\mathbf{m}(x,t) = (m_1(x,t), m_2(x,t), m_3(x,t))$,

at position $x \in [0, L]$ and time $t \geq 0$ in a long thin ferromagnetic material of length $L > 0$. If only the exchange energy term is considered, the magnetization is modelled by the one-dimensional (uncontrolled) Landau–Lifshitz equation (Brown, 1963), (Guo & Ding, 2008, Chapter 6)

$$\frac{\partial \mathbf{m}}{\partial t} = \mathbf{m} \times \mathbf{m}_{xx} - \nu \mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx}) \tag{1a}$$

$$\mathbf{m}(x,0) = \mathbf{m}_0(x) \tag{1b}$$

where $\times$ denotes the cross product and $\nu \geq 0$ is the damping parameter, which depends on the type of ferromagnet. The term $\mathbf{m}_{xx}$ denotes magnetization differentiated with respect to x twice. The Landau–Lifshitz equation sometimes includes a parameter called the gyromagnetic ratio multiplying $\mathbf{m} \times \mathbf{m}_{xx}$. The gyromagnetic ratio has been set to 1 for simplicity. For more on the damping parameter and gyromagnetic ratio, see Gilbert (2004).

The Landau–Lifshitz equation is a coupled set of three nonlinear PDEs. It is assumed that there is no magnetic flux at the boundaries and so Neumann boundary conditions are appropriate:

$$\mathbf{m}_x(0,t) = \mathbf{m}_x(L,t) = \mathbf{0}. \tag{1c}$$

Existence and uniqueness of solutions to (1) with different degrees of regularity has been shown (Alouges & Soyeur, 1992; Carbou & Fabrie, 2001).

Theorem 1 (*Guo & Ding, 2008*, *Lemma 6.3.1*). *If* $\|\mathbf{m}_0(x)\|_2 = 1$, *the solution,* $\mathbf{m}$, *to* (1a) *satisfies*

$$\|\mathbf{m}(x,t)\|_2 = 1 \tag{2}$$

where $\|\cdot\|_2$ *is the Euclidean norm.*

The following statement is a more restrictive version of the theorem stated in Carbou and Fabrie (2001).

Theorem 2 (*Carbou & Fabrie, 2001*, *Thm. 1.3,1.4*). *If* $\mathbf{m}_0 \in H_2(0,L)$, $\mathbf{m}_{0,x}(0) = \mathbf{m}_{0,x}(L) = \mathbf{0}$ *and* $\|\mathbf{m}_0\|_2 = 1$, *then there exist a time* $T^* > 0$ *and a unique solution* $\mathbf{m}$ *of* (1) *such that for all* $T < T^*$, $\mathbf{m} \in \mathcal{C}(0,T; H_2(0,L)) \cap \mathcal{L}_2(0,L; H_3(0,L))$.

With more general initial conditions, solutions to (1) are defined on $\mathcal{L}_2^3 = \mathcal{L}_2([0,L]; \mathbb{R}^3)$ with the usual inner-product and norm. The notation $\|\cdot\|_{\mathcal{L}_2^3}$ is used for the norm. Define the operator

$$f(\mathbf{m}) = \mathbf{m} \times \mathbf{m}_{xx} - \nu \mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx}), \tag{3}$$

and its domain

$$D = \{\mathbf{m} \in \mathcal{L}_2^3 : \mathbf{m}_x \in \mathcal{L}_2^3,\ \mathbf{m}_{xx} \in \mathcal{L}_2^3,\ \mathbf{m}_x(0) = \mathbf{m}_x(L) = \mathbf{0}\}. \tag{4}$$

Theorem 3 (*Chow, 2013*, *Theorem 4.7*). *The operator* $f(\mathbf{m})$ *with domain D generates a nonlinear contraction semigroup on* $\mathcal{L}_2^3$.

Ferromagnets are magnetized to saturation (Cullity & Graham, 2009, Section 4.1); that is $\|\mathbf{m}_0(x)\|_2 = M_s$ where M_s is the magnetization saturation. In much of the literature, M_s is set to 1; see for example, Alouges and Soyeur (1992), Carbou and Fabrie (2001), Guo and Ding (2008, Section 6.3.1) and Lakshmanan (2011). This convention is used here. Physically, this means that at each point, x, the magnitude of $\mathbf{m}_0(x)$ equals the magnetization saturation. The initial condition $\mathbf{m}_0(x)$ is furthermore assumed to be real-valued, and hence $\mathbf{m}(x,t)$ for $t > 0$ is real-valued.

The set of equilibrium points of (1) is (Guo & Ding, 2008, Theorem 6.1.1)

$$E = \{\mathbf{a} = (a_1, a_2, a_3) : a_1, a_2, a_3 \text{ constants and } \mathbf{a}^{\mathrm{T}}\mathbf{a} = 1\}. \tag{5}$$

Theorem 4 (*Chow, 2013*, *Theorem 4.11*). *The equilibrium set in* (5) *is asymptotically stable in the* $\mathcal{L}_2^3$*-norm.*

3. Controller design

In current applications, the control enters as an applied magnetic field (Carbou & Labbé, 2006a,b; Carbou & Labbe, 2012; Carbou et al., 2008, 2009). More precisely, a control, $\mathbf{u}(t)$, is introduced into the Landau–Lifshitz equation (1a) as follows

$$\begin{aligned}\frac{\partial \mathbf{m}}{\partial t} &= \mathbf{m} \times (\mathbf{m}_{xx} + \mathbf{u}) - \nu \mathbf{m} \times (\mathbf{m} \times (\mathbf{m}_{xx} + \mathbf{u})) \\ &= \mathbf{m} \times \mathbf{m}_{xx} - \nu \mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx}) \\ &\quad + \mathbf{m} \times \mathbf{u} - \nu \mathbf{m} \times (\mathbf{m} \times \mathbf{u}),\end{aligned} \tag{6}$$

$$\mathbf{m}(x,0) = \mathbf{m}_0(x).$$

As for the uncontrolled system, the boundary conditions are $\mathbf{m}_x(0,t) = \mathbf{m}_x(L,t) = \mathbf{0}$. Eq. (6) is the Landau–Lifshitz equation with a nonlinear control. Its existence and uniqueness results can be found in Carbou and Fabrie (2001, Thm. 1.1, 1.2) and are similar to Theorem 2.

As for the uncontrolled equation, since

$$\begin{aligned}\frac{1}{2}\frac{\partial \|\mathbf{m}(x,t)\|_2}{\partial t} &= \mathbf{m}^{\mathrm{T}}\frac{\partial \mathbf{m}}{\partial t} \\ &= \mathbf{m}^{\mathrm{T}}(\mathbf{m} \times \mathbf{m}_{xx} - \nu \mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx}) \\ &\quad + \mathbf{m} \times \mathbf{u} - \nu \mathbf{m} \times (\mathbf{m} \times \mathbf{u})) = 0,\end{aligned}$$

this implies $\|\mathbf{m}\|_2 = c$, where c is a constant. The convention is to take $c = 1$. It follows that any equilibrium point is trivially stable in the $\mathcal{L}_2$-norm.

The goal is to choose a control so that the system governed by the Landau–Lifshitz equation moves from an arbitrary initial condition, possibly an equilibrium point, to a specified equilibrium point $\mathbf{r}$. The control needs to be chosen so that $\mathbf{r}$ becomes a stable equilibrium point of the controlled system. It can be shown that zero is an eigenvalue of the linearized uncontrolled Landau–Lifshitz equation (Chow, 2013, Chapter 4.3.2). For finite-dimensional linear systems, simple proportional control of a system with a zero eigenvalue yields asymptotic tracking of a specified state and this motivates choosing the control

$$\mathbf{u} = k(\mathbf{r} - \mathbf{m}) \tag{7}$$

where $\mathbf{r} \in E$ is an equilibrium point of the uncontrolled equation (1) and k is a positive constant control parameter.

Theorem 5. *For any* $\mathbf{r} \in E$ *and any positive constant* k *with control defined in* (7), $\mathbf{r}$ *is a locally stable equilibrium point of* (6) *in the* H_1*-norm. That is, for any initial condition* $\mathbf{m}_0(x) \in D$, *where D is defined in* (4), *the* H_1*-norm of the error* $\mathbf{m} - \mathbf{r}$ *does not increase.*

Proof. Let $B(\mathbf{r}, p) = \{\mathbf{m} \in \mathcal{L}_2^3 : \|\mathbf{m} - \mathbf{r}\|_{\mathcal{L}_2^3} < p\} \subset D$ for some constant $0 < p < 2$. Note that since $p < 2$, then $-\mathbf{r} \notin B(\mathbf{r}, p)$. For any $\mathbf{m} \in B(\mathbf{r}, p)$, consider the H_1-norm of the error

$$V(\mathbf{m}) = k\|\mathbf{m} - \mathbf{r}\|_{\mathcal{L}_2^3}^2 + \|\mathbf{m}_x\|_{\mathcal{L}_2^3}^2.$$

Taking the derivative of V,

$$\begin{aligned}\frac{dV}{dt} &= \int_0^L k(\mathbf{m} - \mathbf{r})^{\mathrm{T}}\dot{\mathbf{m}}dx + \int_0^L \mathbf{m}_x^{\mathrm{T}}\dot{\mathbf{m}}_x dx \\ &= \int_0^L k(\mathbf{m} - \mathbf{r})^{\mathrm{T}}\dot{\mathbf{m}}dx - \int_0^L \mathbf{m}_{xx}^{\mathrm{T}}\dot{\mathbf{m}}dx \\ &= \int_0^L \left(k(\mathbf{m} - \mathbf{r})^{\mathrm{T}}\dot{\mathbf{m}} - \mathbf{m}_{xx}^{\mathrm{T}}\dot{\mathbf{m}}\right)dx.\end{aligned} \tag{8}$$

Let $\mathbf{h} = \mathbf{m} - \mathbf{r}$, then the integrand becomes

$$k\mathbf{h}^{\mathrm{T}}\dot{\mathbf{m}} - \mathbf{m}_{xx}^{\mathrm{T}}\dot{\mathbf{m}} \tag{9}$$