Automatica 67 (2016) 211–215

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

A note on stability of functional difference equations ☆

Daniel Melchor-Aguilar

Division of Applied Mathematics, IPICYT, 78216, San Luis Potosí, SLP, Mexico

ARTICLE INFO

Article history:
Received 6 September 2015
Received in revised form
4 December 2015
Accepted 7 January 2016
Available online 5 February 2016

Keywords:
Functional difference equations
Linear and nonlinear perturbations
Exponential stability

ABSTRACT

In this note, we consider perturbed linear functional difference equations with discrete and distributed delays which play a fundamental role in several stability and stabilizability problems of time-delay systems. Linear and nonlinear perturbations are considered. Sufficient conditions for exponential stability of the perturbed solutions are given.

1. Introduction and problem formulation

Consider the following Functional Difference Equation (FDE):

$$x(t)=\sum_{j=1}^{m}A_jx(t-h_j)+\int_{-\tau}^{0}G(\theta)x(t+\theta)d\theta,\quad t\geq 0,\tag{1}$$

where $A_j\in\mathbb{R}^{n\times n}, 0<h_1<\cdots<h_m, 0<\tau$, and the matrix function $G(\theta)$ has piecewise continuous differentiable elements for $\theta\in[-\tau,0]$.

The stability analysis of equations of the form (1) has recently received a considerable attention (Gil' & Cheng, 2007; Karafyllis & Krstic, 2014; Melchor-Aguilar, Kharitonov, & Lozano, 2010; Melchor-Aguilar, 2010, 2012, 2013a,b; Pepe, 2003, 2014; Shaikhet, 2004, 2011; Verriest, 2001). For specific application problems where systems of the form (1) can be found to see the citations therein of these references. In particular, they are essential in the closed-loop stability of predictor-based state feedbacks for stabilization of systems with input delays (Krstic, 2009).

These recent contributions have shown that new type of results is required in order to properly address the asymptotic behavior of discontinuous solutions of FDEs of the form (1).

For the linear FDE (1) there are still several interesting problems to be addressed. Specifically, the problem of deriving asymptotic stability results for perturbed equations involving general linear and/or nonlinear perturbation terms with discrete and distributed delays has not been sufficiently investigated. As it is well-known, stability results for perturbed linear differential delay equations follow from the variation-of-constants formula which express the solutions of perturbed equations in terms of the fundamental matrix associated to the linear equation. By using such formula to obtain stability results for perturbed linear differential delay equations reduces the discussion to arguments very similar to the ones for ordinary differential equations, see Bellman and Cooke (1963), Hale and Verduyn-Lunel (1993).

By following the definition of a Neutral Functional Differential Equation (NFDE) in the Hale's form, one may be tempted to transform the FDE (1) into the NFDE one

$$\frac{d}{dt}\left[z(t)-\sum_{j=1}^{m}A_jz(t-h_j)\right]=G(0)z(t)$$
$$-G(-\tau)z(t-\tau)-\int_{-\tau}^{0}\dot{G}(\theta)z(t+\theta)d\theta,\tag{2}$$

and then apply the existing results for perturbed linear NFDEs. The problem is that the fundamental matrix $X(t)$, solution of the differential matrix equation

$$\frac{d}{dt}\left[X(t)-\sum_{j=1}^{m}A_jX(t-h_j)\right]=G(0)X(t)$$
$$-G(-\tau)X(t-\tau)-\int_{-\tau}^{0}\dot{G}(\theta)X(t+\theta)d\theta,$$

with initial condition $X(t)=0, t<0$, and $X(0)=I$, cannot admit an exponentially decreasing upper bound. The lack of an exponential decreasing upper bound for $X(t)$ follows from

☆ The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Rafael Vazquez under the direction of Editor Miroslav Krstic.

E-mail address: dmelchor@ipicyt.edu.mx.

http://dx.doi.org/10.1016/j.automatica.2016.01.049
0005-1098/

Theorem 3.2 in page 271 of Hale and Verduyn-Lunel (1993) and the fact that $s = 0$ belongs to the spectrum of the NFDE (2). As a consequence, the asymptotic stability problem of a perturbed FDE (1) cannot be solved by using the existing results for NFDEs.

The goal of this note is to present some results for such a problem. We firstly present an appropriate definition of the fundamental matrix that allows the description of piecewise continuous solutions of (1) and could admit an exponentially decreasing upper bound. Then, we obtain the variation-of-constants formula for nonhomogeneous equations and use it for deriving asymptotic stability results of perturbed equations involving general linear and/or nonlinear perturbation terms with discrete and distributed delays.

2. Main results

We recall that the spectrum of (1) consists of all zeros of the characteristic function

$$f(s) = \det\left(I - \sum_{j=1}^{m} A_j e^{-h_j s} - \int_{-\tau}^{0} G(\theta) e^{s\theta} d\theta\right).$$

Let us suppose that the spectrum of (1) does not contain the point $s = 0$. Then the matrix

$$K_0 = \left(I - \sum_{j=1}^{m} A_j - \int_{-\tau}^{0} G(\theta) d\theta\right)^{-1}$$

is well-defined. Let matrix $K(t)$ be the unique solution of the matrix equation

$$K(t) = \sum_{j=1}^{m} A_j K(t - h_j) + \int_{-\tau}^{0} G(\theta) K(t + \theta) d\theta, \quad t \geq 0, \tag{3}$$

with the initial condition $K(t) = -K_0$, $t \in [-r, 0)$, where $r = \max\{h_m, \tau\}$. The matrix function $K(t)$ is piecewise continuous and of bounded variation.

Next, consider the nonhomogeneous FDE

$$x(t) = \sum_{j=1}^{m} A_j x(t - h_j) + \int_{-\tau}^{0} G(\theta) x(t + \theta) d\theta + w(t), \tag{4}$$

$$x(t) = \varphi(t), \quad t \in [-r, 0),$$

where $w(t)$ is any given piecewise continuous vector function defined for all $t \geq 0$. We assume that the initial function φ belongs to the space of piecewise continuous vector functions $\mathcal{PC} = \mathcal{PC}([-r, 0), \mathbb{R}^n)$ equipped with the standard uniform norm $\|\varphi\|_r = \sup_{\theta \in [-r,0)} \|\varphi(\theta)\|$.

It is worth noting that although more general functional spaces than $\mathcal{PC}$ can be considered as, for instance, $\mathcal{L}^p$ spaces (Carvalho, 1996; Gil' & Cheng, 2007; Karafyllis & Krstic, 2014), the $\mathcal{PC}$ space suffices the technical proofs of our results and, at the same time, it is sufficiently general for the stability investigation of the kind of problems mentioned in the introduction.

Lemma 1. *If $s = 0$ does not belong to the spectrum of (1) then for any given initial function $\varphi \in \mathcal{PC}$ the corresponding solution $x(t, \varphi)$ of (4) can be represented as*

$$x(t, \varphi) = \int_{-\tau}^{0} \left(\int_{-\tau}^{\xi} [d_\theta K(t + \theta - \xi)] G(\theta)\right) \varphi(\xi) d\xi - \sum_{j=1}^{m} \int_{-h_j}^{0} \left[d_\xi K(t - h_j - \xi)\right] A_j \varphi(\xi) - \int_{0}^{t+0} \left[d_\xi K(t - \xi)\right] w(\xi), \quad t \geq 0, \tag{5}$$

where the integrals are Riemann–Stieltjes ones.

Proof. By computing the Laplace image on both sides of (4) one gets

$$\hat{x}(s) = H(s) \sum_{j=1}^{m} A_j \int_{-h_j}^{0} \varphi(\xi) e^{-s(\xi + h_j)} d\xi + H(s) \hat{w}(s) + H(s) \int_{-\tau}^{0} \left(\int_{-\tau}^{\xi} G(\theta) e^{s\theta} d\theta\right) \varphi(\xi) e^{-s\xi} d\xi, \tag{6}$$

where $\hat{x}(s)$ and $\hat{w}(s)$ respectively denote the Laplace image of $x(t)$ and $w(t)$, and

$$H(s) = \left(I - \sum_{j=1}^{m} A_j e^{-h_j s} - \int_{-\tau}^{0} G(\theta) e^{s\theta} d\theta\right)^{-1}.$$

From (3) one gets that the Laplace image $\hat{K}(s)$, of the matrix $K(t)$, satisfies

$$H(s) = s\hat{K}(s) + K_0.$$

Taking into account this equation and observing that the Laplace image of $\frac{d}{dt}K(t)$ is equal to $s\hat{K}(s) - K(0) = s\hat{K}(s) - (I - K_0)$, an application of the convolution theorem on the right-hand side of (6) yields

$$x(t, \varphi) = \sum_{j=1}^{m} \int_{-h_j}^{0} \left[\frac{\partial K(t - h_j - \xi)}{\partial t}\right] A_j \varphi(\xi) + \sum_{j=1}^{m} A_j \varphi(t - h_j) q(t - h_j) + \int_{-\tau}^{0} \left(\int_{-\tau}^{\xi} \left[\frac{\partial K(t + \theta - \xi)}{\partial t}\right] G(\theta)\right) \varphi(\xi) d\xi + \int_{t-\tau}^{t} G(\xi - t) \varphi(\xi) q(\xi) d\xi + \left[\int_{0}^{t} \frac{\partial K(t - \xi)}{\partial t}\right] w(\xi) d\xi + w(t), \tag{7}$$

where $q(\xi) = 1$ for $\xi < 0$ and $q(\xi) = 0$ for $\xi \geq 0$. Consider the term $A_j \varphi(t - h_j) q(t - h_j)$. If $t \geq h_j$ this term is not present. If $t < h_j$ this term is $A_j \varphi(t - h_j)$, which is precisely the Riemann–Stieltjes integral

$$-\int_{0}^{t-h_j} \left[d_\xi K(t - h_j - \xi)\right] A_j \varphi(\xi).$$

Similarly, if $t \geq \tau$ then the integral term $\int_{t-\tau}^{t} G(\xi - t) \varphi(\xi) q(\xi) d\xi$ is not present while if $t < \tau$ then the term is $\int_{t-\tau}^{0} G(\xi - t) \varphi(\xi) d\theta$, which is precisely the Riemann–Stieltjes integral

$$\int_{t-\tau}^{0} \left(\int_{-\tau}^{\xi - t} [d_\theta K(t + \theta - \xi)] G(\theta)\right) \varphi(\xi) d\xi.$$

Also, $w(t)$ is the value of the Riemann–Stieltjes integral

$$-\int_{t}^{t+0} d_\xi K(t - \xi) w(\xi).$$

Therefore, if we make use of the Riemann–Stieltjes integral, the expression for $x(t, \varphi)$ given by (7) can be written as (5) and, thus, the lemma is proved. ■

Remark 1. The formula (5) comprises some existing ones for describing piecewise continuous solutions of FDEs of the form (1). For instance, in the particular case of (1) when $m = 1$, $A = A_1$, $h =$