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Brief paper

Further results on identifiability of discrete-time nonlinear systems ☆

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ABSTRACT

Different notions of identifiability are available for discrete-time nonlinear systems. The most popular in the current literature is structural identifiability, besides algebraic identifiability and identifiability with known initial conditions. These three notions are fully characterized in this paper within a unique linear algebraic framework. The relationships between the notions are shown as well. Academic examples as well as the model of a power amplifier, borrowed from the area of digital telecommunications, are used to illustrate the different notions and computations involved by our results.

1. Introduction

Discrete time systems are a popular choice in modeling different real life processes. Answering the question whether or not the parameters of the system's mathematical model are uniquely determined from the input–output data, *i.e.* checking identifiability, is mandatory before running identification algorithms. While the problem of parameter identification has been extensively treated for the both continuous- and discrete-time cases, surprisingly identifiability has got a systematic treatment for the continuous-time case only (Grewal & Glover, 1976; Lecourtier, Lamnabhi-Lagarrigue, & Walter, 1987; Ljung, 1986; Saccomani, Audoly, & D'Angi, 2003; Tunali & Tarn, 1987; Walter & Pronzato, 1996; Xia & Moog, 2003). The literature about identifiability of discrete time systems is much less numerous (Dasgupta, Shrivastava, & Krenzer, 1991; Soderstrom, Ljung, & Gustavsson, 1976; Van den Hof, 1998). Characterizations of identifiability of linear discrete time-systems can be found in Dötsch and Van Den Hof (1996) and Glover and Willems (1974). The local state isomorphism approach has been applied to characterize structural identifiably of discrete-time nonlinear systems (Anstett, Bloch, Millérioux, & Denis-Vidal, 2008). Computational aspects, and the possibility to check algebraic and structural identifiability properties of the given model with computer algebra system *Mathematica* has been studied in Tabun, Nõmm, Kotta, and Moog (2006).

The concept of so called *structural identifiability* (Tunali & Tarn, 1987), is the most studied one but its relations to the other concepts are not properly established. This leads to the main goals of this paper, to give proper characterizations to the different notions of identifiability and describe relations between them. Current studies are focused on the following identifiability concepts: structural, algebraic, algebraic with known initial conditions. For this purpose the linear algebraic framework introduced in Grizzle (1993) will be used. It will be shown that, with suitable mathematical tools, the results of Xia and Moog (2003) can be extended generically to discrete-time case. Some preliminary results on the identifiability of nonlinear discrete-time systems have been published in Nõmm and Moog (2004).

The paper is organized is as follows. Section 2 recalls the algebraic framework which is used. Section 3 presents the definitions of the different notions of identifiability for discrete-time nonlinear systems, illustrated either by academic examples or by a model of amplifier borrowed from telecommunications. The main results on the characterization of different notions of identifiability given in Section 4 as well as their relationships. Section 5 gives a short overview of mainstream about identifiability for discrete-time linear systems. They are shown to be consistent with the definition of algebraic identifiability. Section 6 draws some conclusions.

2. Algebraic framework

The linear algebraic framework introduced in Aranda-Bricaire, Kotta, and Moog (1996) and Grizzle (1993) is used throughout

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this paper and is shown to be able to unify various concepts of identifiability. The basic notations, definitions and main tools are recalled now. Consider a discrete-time nonlinear system described by its state-space equations.

$$\begin{aligned} x(t+1) &= f\big(x(t),\theta,u(t)\big), \quad x_0 = x(0) \\ y(t) &= h\big(x(t),\theta\big), \end{aligned} \tag{1}$$

where $x(t) \in X = \mathbb{R}^n$ is the state, $u(t) \in U = \mathbb{R}^m$ is the control input and, $y(t) \in Y = \mathbb{R}^p$, is the measured output. The functions $f(x(t),\theta,u(t))$ and $h(x(t),\theta,u(t))$ are analytic functions of their arguments on a connected open set $\mathcal{M} \subset \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^m$, finally $\theta \in \mathcal{P} \subset \mathbb{R}^q$ is a vector of parameters to be identified. Let $\mathcal{R}_k$ denote the ring of real analytic functions of variables $\{x(0),\theta,u(0),\dots,u(k)\}$ and $\mathcal{K}_k$ denote the associated field of meromorphic functions of a finite number of variables $\{x(0),\theta,u(0),\dots,u(k)\}$. A typical element of $\mathcal{K}_k$ is $\eta(v) = \nu(v)/\xi(v)$ where ν and ξ are analytic functions and $v = (v_1,\dots,v_j)$ denotes various components of $\{x(0),\theta,u(0),\dots,u(k)\}$. The formal differential of η is

$$d\eta(v) = \sum_{i=1}^{j} \frac{\partial \eta(v)}{\partial v_i} dv_i. \tag{2}$$

Define

$$\mathcal{K} = \bigcup_{k\geq 0} \mathcal{K}_k, \qquad d\mathcal{K} = \{d\varphi \mid \varphi \in \mathcal{K}\} \tag{3}$$

and following the lines of Grizzle (1993) let $\mathcal{E}$ denote the vector space over $\mathcal{K}$ spanned by $\{\mathrm{d}x(0), \mathrm{d}\theta, \mathrm{d}u(0),\dots,\mathrm{d}u(k),\dots\}$. More formally,

$$\mathcal{E} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\mathcal{K}\}. \tag{4}$$

The elements of $\mathcal{E}$ are called differential one forms. The following are special subspaces of vector space $\mathcal{E}$.

$$\begin{aligned} \mathcal{Y}_N &= \bigcup_{k=0}^{N} \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y(0),\dots,\mathrm{d}y(k)\}, \\ \mathcal{X} &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x(0)\} \\ \mathcal{U}_N &= \bigcup_{k=0}^{N} \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u(0),\dots,\mathrm{d}u(k)\}, \\ \Theta &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\theta\}. \end{aligned} \tag{5}$$

Here $\mathcal{X}$ represents the state-space, $\mathcal{Y}_N$ represents the output space for time $t = 1,\dots,N$, $\mathcal{U}_N$ represents the input space for time $t = 1,\dots,N$ and Θ represents the space of parameters. More details about the algebraic framework and its application to different control problems are found in Aranda-Bricaire et al. (1996) and Grizzle (1993).

3. Different concepts of identifiability

Consider a discrete-time nonlinear system (1). The following definition of local identifiability around some point is well known in the continuous time case (Tunali & Tarn, 1987) and is adapted for the discrete-time case as follows.

Definition 1. System (1) is said to be locally strongly x_0-identifiable at θ through the input sequence $\{u(t)\}_0^T$ if there exists an open neighborhood $\mathcal{P}^0$ of θ: $\mathcal{P}^0 \subset \mathcal{P}$ such that for any $\theta_1, \theta_2 \in \mathcal{P}^0$

$$\theta_1 \neq \theta_2 \Rightarrow \{y(t,\theta_1,x_0,u(t))\}_0^T \not\equiv \{y(t,\theta_2,x_0,u(t))\}_0^T. \tag{6}$$

The following academic example is not identifiable at any point x_0.

Example 2.

$$\begin{aligned} x_1(t+1) &= x_2(t) + u(t) \\ x_2(t+1) &= \theta_1 + \theta_2 \\ y(t) &= x_1(t). \end{aligned} \tag{7}$$

This leads $y(t) = \theta_1 + \theta_2 - u(t-1)\ \forall t \geq 1$. Thus, one single independent equation is derived for the two independent unknown parameters θ_1 and θ_2. The latter cannot be uniquely computed and the system (7) is not identifiable in the sense of Definition 1 around any point x_0.

To introduce the weaker notion of structural identifiability, it is required to introduce a topology for the space of all input sequences. A topology of this space is associated to the following norm

$$\|\{r(t)\}_0^T\| = \sqrt{\sum_{i=0}^{T} |r(i)|^2}. \tag{8}$$

For any positive T denote by U^T the space of all input sequences. The space of the corresponding output sequences is denoted Y^T.

Definition 3. System (1) is said to be structurally identifiable if there exist a $T > 0$ and open and dense subsets $\mathcal{M}_0 \subset \mathcal{M}$, $\mathcal{P}^0 \subset \mathcal{P}$ and $U_0^T \subset U^T$ such that the system Σ_θ is locally strongly x_0-identifiable at θ through the input sequence $\{u(t)\}_0^T$ for every $x_0 \in \mathcal{M}_0$, $\theta \in \mathcal{P}^0$ and $\{u(t)\}_0^T \in U_0^T$.

The notion of structural identifiability corresponds to local identifiability around any point x_0 of an open and dense subset M_0. Whereas Definition 1 stands for local identifiability, Definition 3 denotes a generic identifiability (almost everywhere).

To illustrate Definitions 1 and 3, consider the following example.

Example 4.

$$\begin{aligned} x(t+1) &= \theta x^3(t) \\ y(t) &= x(t). \end{aligned} \tag{9}$$

For $x_0 = 0, \theta_1 \neq \theta_2$ does not imply that $y(t,\theta_1,0) \neq y(t,\theta_2,0)$. Consequently system (9) is not 0-identifiable according to Definition 1. At the same time for $x_0 \neq 0$, relation (6) holds, which means, that the system (9) is x_0-identifiable. In terms of Definition 3 the system (9) is structurally identifiable as $\mathcal{M}_0 = \mathbb{R} \setminus \{0\}$ is open and dense in $\mathbb{R}$.

Identifiability in the sense of Definitions 1 and 3 requires the knowledge of the initial condition x_0. The following notion of algebraic identifiability is much stronger, since the computation of the parameters has to be enabled independently from the knowledge of the initial condition.

Definition 5. System (1) is said to be algebraically identifiable if there exist a positive integer T, open and dense subsets $\mathcal{M}_0 \subset \mathcal{M}$, $\mathcal{P}^0 \subset \mathcal{P}$, $U_0^T \subset U^T$ and a meromorphic function $\Phi : \mathbb{R}^q \times \mathbb{R}^{(T+1)p} \times \mathbb{R}^{Tm} \to \mathbb{R}^q$ such that:

$$\mathrm{rank}\frac{\partial \Phi}{\partial \theta} = q \tag{10}$$

and

$$\Phi(\theta, y(0),\dots,y(T),u(0),\dots,u(T-1)) = 0 \tag{11}$$

for every $(\theta, y(0),\dots,y(T),u(0),\dots,u(T-1))$, satisfying the dynamics of the system, and $(\theta,u(0),\dots,u(T-1)) \in \mathcal{P}^0 \times \mathcal{M}_0 \times U_0^T$.