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Gaussian filter for nonlinear systems with correlated noises at the same epoch[☆]

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ABSTRACT

This paper proposes a general framework solution of Gaussian filter (GF) for both linear and nonlinear dynamic systems with correlated noises at the same epoch. Detailed discussions and simulation comparisons with existing Gaussian approximation recursive filter and existing de-correlating GF are provided, which show advantages of estimation accuracy of the proposed method in some applications.

1. Introduction

Nonlinear filtering with correlated noises at the same epoch has been gaining more attention in many applications. There are two frameworks available to solve this problem including the de-correlating framework (Bar-Shalom, Li, & Kirubarajan, 2001) and Gaussian approximation recursive filter (GASF) framework (Wang, Liang, Pan, & Yang, 2012). For linear Gaussian system, these two frameworks are completely equivalent (Chang, 2014a), and for general nonlinear system, they are equivalent in the linear minimum mean square error (MMSE) sense (Wang, Liang, Pan, & Wang, 2014). Many nonlinear filters have been derived based on these two frameworks (Chang, 2014b; Chen & Ma, 2011; Xu, Dimirovski, Jing, & Shen, 2007). However, both GASF and de-correlating frameworks have some drawbacks in the application of nonlinear systems, as will be discussed in Sections 3 and 4. In this paper, a novel general framework of correlated Gaussian approximate filter (CGAF) for nonlinear systems with correlated noises at the same epoch is derived and compared with existing methods.

2. General framework of CGAF

Consider the following discrete-time nonlinear stochastic system with correlated noises at the same epoch as shown by the state-space model

$$\begin{cases} \boldsymbol{x}_k = \boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1}) + \boldsymbol{G}_{k-1}\boldsymbol{w}_{k-1} \\ \boldsymbol{z}_k = \boldsymbol{h}_k(\boldsymbol{x}_k) + \boldsymbol{v}_k \end{cases} \tag{1}$$

where k is the discrete time, $\boldsymbol{f}_{k-1}(\cdot)$ and $\boldsymbol{h}_k(\cdot)$ are some known functions, $\boldsymbol{G}_{k-1} \in \mathbb{R}^{n\times q}$ is known process noise matrix, $\boldsymbol{x}_k \in \mathbb{R}^n$ is the state vector, $\boldsymbol{z}_k \in \mathbb{R}^m$ is the measurements vector, $\boldsymbol{w}_k \in \mathbb{R}^q$ and $\boldsymbol{v}_k \in \mathbb{R}^m$ are correlated zero-mean Gaussian white noises satisfying $E[\boldsymbol{w}_k\boldsymbol{w}_l^T] = \boldsymbol{Q}_k\delta_{kl}$, $E[\boldsymbol{v}_k\boldsymbol{v}_l^T] = \boldsymbol{R}_k\delta_{kl}$ and $E[\boldsymbol{w}_k\boldsymbol{v}_l^T] = \boldsymbol{S}_k\delta_{kl}$, where δ_{kl} is the Kronecker delta function, the initial state $\boldsymbol{x}_0$ is a Gaussian random vector with mean $\hat{\boldsymbol{x}}_{0|0}$ and covariance $\boldsymbol{P}_{0|0}$, and it is uncorrelated with $\boldsymbol{w}_k$ and $\boldsymbol{v}_k$. Similar to that in Arasaratnam and Haykin (2009), we present a Gaussian assumption which has been widely accepted and used to design the CGAF for system formulated in (1).

Assumption 1. $\boldsymbol{x}_k$ and $\boldsymbol{z}_k$ are jointly Gaussian conditioned on previous measurements $\boldsymbol{Z}_{k-1}$, i.e. $p(\boldsymbol{x}_k, \boldsymbol{z}_k|\boldsymbol{Z}_{k-1})$ is Gaussian, where $\boldsymbol{Z}_{k-1} = \{\boldsymbol{z}_j\}_{j=1}^{k-1}$.

The Gaussian assumption is reasonable for some applications with mild nonlinearity, such as target tracking (Arasaratnam & Haykin, 2009; Bar-Shalom et al., 2001) and ballistic target reentry (Chang, 2014b).

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Similar to that in Wang et al. (2012), in the case that process noise and measurement noise are correlated at the same epoch, we need to define the following augmented state vector $\boldsymbol{\xi}_k = [\boldsymbol{x}_k^T \quad \boldsymbol{w}_k^T]^T$. If Gaussian approximations to $p(\boldsymbol{x}_k|\boldsymbol{Z}_k)$ and $p(\boldsymbol{w}_k|\boldsymbol{Z}_k)$ have been updated, the posterior probability density function (PDF) $p(\boldsymbol{\xi}_k|\boldsymbol{Z}_k)$ of the augmented state $\boldsymbol{\xi}_k$ is also Gaussian, and its first two moments can be formulated as

$$\begin{cases} \hat{\boldsymbol{\xi}}_{k|k} = \begin{bmatrix} \hat{\boldsymbol{x}}_{k|k} \\ \hat{\boldsymbol{w}}_{k|k} \end{bmatrix} \boldsymbol{P}^{\xi\xi}_{k|k} = \begin{bmatrix} \boldsymbol{P}_{k|k} & \boldsymbol{P}^{xw}_{k|k} \\ (\boldsymbol{P}^{xw}_{k|k})^T & \boldsymbol{P}^{ww}_{k|k} \end{bmatrix} \\ \boldsymbol{P}^{xw}_{k|k} = E[\tilde{\boldsymbol{x}}_{k|k}\tilde{\boldsymbol{w}}^T_{k|k}|\boldsymbol{Z}_k] \end{cases} \tag{2}$$

where $\tilde{\boldsymbol{x}}_{k|k} = \boldsymbol{x}_k - \hat{\boldsymbol{x}}_{k|k}$ and $\tilde{\boldsymbol{w}}_{k|k} = \boldsymbol{w}_k - \hat{\boldsymbol{w}}_{k|k}$.

Theorem 1. *Based on Assumption 1, the Gaussian approximation of $p(\boldsymbol{x}_k|\boldsymbol{Z}_k)$ has the filtering estimation $\hat{\boldsymbol{x}}_{k|k}$ and the covariance $\boldsymbol{P}_{k|k}$ at time k of the state $\boldsymbol{x}_k$ as the unified form:*

$$\hat{\boldsymbol{x}}_{k|k} = \hat{\boldsymbol{x}}_{k|k-1} + \boldsymbol{K}^x_k(\boldsymbol{z}_k - \hat{\boldsymbol{z}}_{k|k-1}) \tag{3}$$

$$\boldsymbol{P}_{k|k} = \boldsymbol{P}_{k|k-1} - \boldsymbol{K}^x_k\boldsymbol{P}^{zz}_{k|k-1}(\boldsymbol{K}^x_k)^T \tag{4}$$

$$\boldsymbol{K}^x_k = \boldsymbol{P}^{xz}_{k|k-1}(\boldsymbol{P}^{zz}_{k|k-1})^{-1} \tag{5}$$

where

$$\hat{\boldsymbol{x}}_{k|k-1} = \int_{\mathbb{R}^n} \boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1})N(\boldsymbol{x}_{k-1}; \hat{\boldsymbol{x}}_{k-1|k-1}, \boldsymbol{P}_{k-1|k-1}) \times d\boldsymbol{x}_{k-1} + \boldsymbol{G}_{k-1}\hat{\boldsymbol{w}}_{k-1|k-1} \tag{6}$$

$$\begin{aligned}\boldsymbol{P}_{k|k-1} = &\int_{\mathbb{R}^n} (\boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1}) + \boldsymbol{G}_{k-1}\hat{\boldsymbol{w}}_{\boldsymbol{x},k-1|k-1}) \\ &\times (\boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1}) + \boldsymbol{G}_{k-1}\hat{\boldsymbol{w}}_{\boldsymbol{x},k-1|k-1})^T \\ &\times N(\boldsymbol{x}_{k-1}; \hat{\boldsymbol{x}}_{k-1|k-1}, \boldsymbol{P}_{k-1|k-1})d\boldsymbol{x}_{k-1} \\ &- \hat{\boldsymbol{x}}_{k|k-1}\hat{\boldsymbol{x}}^T_{k|k-1} + \boldsymbol{G}_{k-1}\boldsymbol{\Omega}_{k-1|k-1}\boldsymbol{G}^T_{k-1}\end{aligned} \tag{7}$$

$$\hat{\boldsymbol{w}}_{\boldsymbol{x},k-1|k-1} = \hat{\boldsymbol{w}}_{k-1|k-1} + (\boldsymbol{P}^{xw}_{k-1|k-1})^T\boldsymbol{P}^{-1}_{k-1|k-1}(\boldsymbol{x}_{k-1} - \hat{\boldsymbol{x}}_{k-1|k-1}) \tag{8}$$

$$\boldsymbol{\Omega}_{k-1|k-1} = \boldsymbol{P}^{ww}_{k-1|k-1} - (\boldsymbol{P}^{xw}_{k-1|k-1})^T\boldsymbol{P}^{-1}_{k-1|k-1}\boldsymbol{P}^{xw}_{k-1|k-1} \tag{9}$$

$$\hat{\boldsymbol{z}}_{k|k-1} = \int_{\mathbb{R}^n} \boldsymbol{h}_k(\boldsymbol{x}_k)N(\boldsymbol{x}_k; \hat{\boldsymbol{x}}_{k|k-1}, \boldsymbol{P}_{k|k-1})d\boldsymbol{x}_k \tag{10}$$

$$\boldsymbol{P}^{zz}_{k|k-1} = \int_{\mathbb{R}^n} \boldsymbol{h}_k(\boldsymbol{x}_k)\boldsymbol{h}^T_k(\boldsymbol{x}_k)N(\boldsymbol{x}_k; \hat{\boldsymbol{x}}_{k|k-1}, \boldsymbol{P}_{k|k-1})d\boldsymbol{x}_k - \hat{\boldsymbol{z}}_{k|k-1}\hat{\boldsymbol{z}}^T_{k|k-1} + \boldsymbol{R}_k \tag{11}$$

$$\boldsymbol{P}^{xz}_{k|k-1} = \int_{\mathbb{R}^n} \boldsymbol{x}_k\boldsymbol{h}^T_k(\boldsymbol{x}_k)N(\boldsymbol{x}_k; \hat{\boldsymbol{x}}_{k|k-1}, \boldsymbol{P}_{k|k-1})d\boldsymbol{x}_k - \hat{\boldsymbol{x}}_{k|k-1}\hat{\boldsymbol{z}}^T_{k|k-1}. \tag{12}$$

Proof. Similar to the proof of the state correction of GASF in Wang et al. (2012), based on Gaussian Assumption 1, the posterior PDF $p(\boldsymbol{x}_k|\boldsymbol{Z}_k)$ is updated as Gaussian, and (3)–(6) and (10)–(12) can be easily proved. The proofs of (7)–(9) are given in Appendix A.

Theorem 2. *Based on Assumption 1, the Gaussian approximation of $p(\boldsymbol{w}_k|\boldsymbol{Z}_k)$ has the filtering estimation $\hat{\boldsymbol{w}}_{k|k}$ and the covariance $\boldsymbol{P}^{ww}_{k|k}$ at time k of the process noise $\boldsymbol{w}_k$ as the unified form:*

$$\hat{\boldsymbol{w}}_{k|k} = \boldsymbol{K}^w_k(\boldsymbol{z}_k - \hat{\boldsymbol{z}}_{k|k-1}) \tag{13}$$

$$\boldsymbol{P}^{ww}_{k|k} = \boldsymbol{Q}_k - \boldsymbol{K}^w_k\boldsymbol{P}^{zz}_{k|k-1}(\boldsymbol{K}^w_k)^T \tag{14}$$

$$\boldsymbol{K}^w_k = \boldsymbol{S}_k(\boldsymbol{P}^{zz}_{k|k-1})^{-1} \tag{15}$$

$$\boldsymbol{P}^{xw}_{k|k} = -\boldsymbol{P}^{xz}_{k|k-1}(\boldsymbol{P}^{zz}_{k|k-1})^{-1}\boldsymbol{S}^T_k. \tag{16}$$

Proof. See Appendix B.

The proposed CGAF framework formulated in Theorems 1 and 2 includes prediction step in (6)–(9) and update step in (3)–(5) and (10)–(16).

3. Comparisons with existing GASF method and de-correlating method

Existing GASF method, de-correlating method and the proposed CGAF method are almost consistent in computational burden and ease of implementation. However, the proposed CGAF framework is different to existing frameworks in assumptions and computations of state prediction estimation and corresponding prediction error covariance matrix. Moreover, the proposed CGAF has advantages as compared with existing GASF for nonlinear process functions and de-correlating method for nonlinear measurement functions in estimation accuracy, as will be discussed in this section and shown in simulations.

(1) It is easy to prove that the proposed CGAF is equivalent with GASF for linear process functions and de-correlating filter for linear measurement functions. Also we can prove that the proposed CGAF is equivalent with standard GF while existing GASF is not equivalent with standard GF for nonlinear systems with nonlinear process functions and uncorrelated noises.

(2) Both the proposed CGAF and de-correlating algorithms are based on Assumption 1, however, GASF algorithm is based on the Gaussianity of $p(\boldsymbol{x}_k, \boldsymbol{z}_{k-1}|\boldsymbol{Z}_{k-2})$.

(3) Substituting (13) and (15) into (6), we can obtain the state prediction estimation $\hat{\boldsymbol{x}}_{k|k-1}$ of the proposed CGAF as follows:

$$\hat{\boldsymbol{x}}_{k|k-1} = E[\boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1})|\boldsymbol{Z}_{k-1}] + \boldsymbol{G}_{k-1}\boldsymbol{S}_{k-1}(\boldsymbol{P}^{zz}_{k-1|k-2})^{-1} \times (\boldsymbol{z}_{k-1} - \hat{\boldsymbol{z}}_{k-1|k-2}) \tag{17}$$

where the expectation $E[\cdot|\boldsymbol{Z}_{k-1}]$ is with respect to PDF $N(\boldsymbol{x}_{k-1}; \hat{\boldsymbol{x}}_{k-1|k-1}, \boldsymbol{P}_{k-1|k-1})$. The $\hat{\boldsymbol{x}}_{k|k-1}$ of GASF is as follows (Wang et al., 2012):

$$\begin{aligned}\hat{\boldsymbol{x}}_{k|k-1} = &E[\boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1})|\boldsymbol{Z}_{k-2}] + \boldsymbol{\psi}(\boldsymbol{P}^{zz}_{k-1|k-2})^{-1}(\boldsymbol{z}_{k-1} - \hat{\boldsymbol{z}}_{k-1|k-2}) \\ &+ \boldsymbol{G}_{k-1}\boldsymbol{S}_{k-1}(\boldsymbol{P}^{zz}_{k-1|k-2})^{-1}(\boldsymbol{z}_{k-1} - \hat{\boldsymbol{z}}_{k-1|k-2})\end{aligned} \tag{18}$$

where the expectation $E[\cdot|\boldsymbol{Z}_{k-2}]$ is with respect to PDF $N(\boldsymbol{x}_{k-1}; \hat{\boldsymbol{x}}_{k-1|k-2}, \boldsymbol{P}_{k-1|k-2})$, $\boldsymbol{\psi} = E[\tilde{\boldsymbol{f}}_{k-1|k-2}(\boldsymbol{x}_{k-1})\tilde{\boldsymbol{h}}^T_{k-1|k-2}(\boldsymbol{x}_{k-1})|\boldsymbol{Z}_{k-2}]$, $\tilde{\boldsymbol{f}}_{k-1|k-2}(\boldsymbol{x}_{k-1}) = \boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1}) - E[\boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1})|\boldsymbol{Z}_{k-2}]$, $\tilde{\boldsymbol{h}}_{k-1|k-2}(\boldsymbol{x}_{k-1}) = \boldsymbol{h}_{k-1}(\boldsymbol{x}_{k-1}) - E[\boldsymbol{h}_{k-1}(\boldsymbol{x}_{k-1})|\boldsymbol{Z}_{k-2}]$. The $\hat{\boldsymbol{x}}_{k|k-1}$ of de-correlating filter is as follows (Chang, 2014b):

$$\hat{\boldsymbol{x}}_{k|k-1} = E[\boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1})|\boldsymbol{Z}_{k-1}] + \boldsymbol{G}_{k-1}\boldsymbol{S}_{k-1}(\boldsymbol{R}_{k-1})^{-1} \times (\boldsymbol{z}_{k-1} - E[\boldsymbol{h}_{k-1}(\boldsymbol{x}_{k-1})|\boldsymbol{Z}_{k-1}]). \tag{19}$$

It can be seen from (17)–(19) that the state prediction estimation of the proposed filter is different to that of existing GASF in computing $E[\boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1})|\boldsymbol{Z}_{k-1}]$ and de-correlating filter in computing $\hat{\boldsymbol{w}}_{k|k}$.

(4) Substituting (8)–(9) and (13)–(16) into (7), the prediction error covariance matrix $\boldsymbol{P}_{k|k-1}$ of the proposed filter can be rewritten as.

$$\begin{aligned}\boldsymbol{P}_{k|k-1} = &E[\tilde{\boldsymbol{f}}_{k-1|k-1}(\boldsymbol{x}_{k-1})\tilde{\boldsymbol{f}}^T_{k-1|k-1}(\boldsymbol{x}_{k-1})|\boldsymbol{Z}_{k-1}] \\ &- \boldsymbol{G}_{k-1}\boldsymbol{S}_{k-1}(\boldsymbol{P}^{zz}_{k-1|k-2})^{-1}(\boldsymbol{P}^{xz}_{k-1|k-2})^T\boldsymbol{P}^{-1}_{k-1|k-1}\boldsymbol{\lambda} \\ &- \boldsymbol{\lambda}^T\boldsymbol{P}^{-1}_{k-1|k-1}\boldsymbol{P}^{xz}_{k-1|k-2}(\boldsymbol{P}^{zz}_{k-1|k-2})^{-1}(\boldsymbol{G}_{k-1}\boldsymbol{S}_{k-1})^T \\ &+ \boldsymbol{G}_{k-1}\boldsymbol{P}^{ww}_{k-1|k-1}\boldsymbol{G}^T_{k-1}\end{aligned} \tag{20}$$

where $\tilde{\boldsymbol{f}}_{k-1|k-1}(\boldsymbol{x}_{k-1}) = \boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1}) - E[\boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1})|\boldsymbol{Z}_{k-1}]$, and $\boldsymbol{\lambda} = E[\tilde{\boldsymbol{x}}_{k-1|k-1}\tilde{\boldsymbol{f}}^T_{k-1|k-1}(\boldsymbol{x}_{k-1})|\boldsymbol{Z}_{k-1}]$. The $\boldsymbol{P}_{k|k-1}$ of GASF can be formulated as (Wang et al., 2012)

$$\begin{aligned}\boldsymbol{P}_{k|k-1} = &\{E[\tilde{\boldsymbol{f}}_{k-1|k-2}(\boldsymbol{x}_{k-1})\tilde{\boldsymbol{f}}^T_{k-1|k-2}(\boldsymbol{x}_{k-1})|\boldsymbol{Z}_{k-2}] \\ &- \boldsymbol{\psi}(\boldsymbol{P}^{zz}_{k-1|k-2})^{-1}\boldsymbol{\psi}^T\} - \boldsymbol{G}_{k-1}\boldsymbol{S}_{k-1}(\boldsymbol{P}^{zz}_{k-1|k-2})^{-1}\boldsymbol{\psi}^T \\ &- \boldsymbol{\psi}(\boldsymbol{P}^{zz}_{k-1|k-2})^{-1}(\boldsymbol{G}_{k-1}\boldsymbol{S}_{k-1})^T + \boldsymbol{G}_{k-1}\boldsymbol{P}^{ww}_{k-1|k-1}\boldsymbol{G}^T_{k-1}.\end{aligned} \tag{21}$$