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Brief paper

A new class of consensus protocols for agent networks with discrete time dynamics[☆]

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ABSTRACT

The paper introduces a new class of consensus protocols to reach an agreement in networks of agents with discrete time dynamics. In order to guarantee the convergence of the proposed algorithms, some general results are proved in the framework of non-negative matrix theory. Moreover, we characterize the set of the consensus protocols and we specify the algorithm that each agent has to employ. Furthermore, we show that in the case of balanced graphs, the agents can apply the consensus protocols by a decentralized and scalable computation. The convergence properties are studied by a set of tests that show the good performance of the proposed algorithm for different network topologies, even in the cases in which the standard protocols do not exhibit satisfying performances. In particular, a rigorous theoretical analysis of the proposed protocol convergence for networks with ring topology is provided and compared with the standard algorithm.

1. Introduction

In recent years the study of the consensus problem has received a great effort by the scientific community involving several fields and many applications. More precisely, in networks of autonomous agents, consensus means to reach an agreement regarding a certain quantity of interest that depends on the state of all the agents. A consensus algorithm (or protocol) is an interaction rule that specifies the information exchange between an agent and all of its neighbors on the network.

In a pioneering contribution, Jadbabaie, Lin, and Morse (2003) provided the theoretical framework for the problem of reaching an agreement on network systems with topology described by undirected graphs. Olfati-Saber and Murray in Olfati-Saber, Fax, and Murray (2007) and Olfati-Saber and Murray (2004) show that the discrete time model of the consensus network is described by a directed or undirected graph and the associated graph Laplacian matrix L plays an important role in the convergence and alignment analysis. Indeed, the nominal state evolution of the agents is governed by a discrete time consensus equation defined as $x(k+1) = (I - \epsilon L)x(k)$, where I is the identity matrix and $\epsilon > 0$ is a stepsize parameter. However, such standard protocols exhibit two drawbacks: the convergence is affected by the choice of the step-size parameter and has a low speed of reaching a consensus for particular graph topologies (i.e., graphs constituted by periodic strong components Diestel, 1999). An alternative form of the standard Laplacian matrix is presented in Fax and Murray (2004), but the proposed algorithm does not converge for periodic graphs.

The convergence speed of consensus protocols is an important topic that has received significant attention in recent years (Fang, Wu, & Wei, 2012). In Xiao and Boyd (2004) the authors find the general conditions to determine the weight to be associated to each node for the linear iteration to converge to the average and to make the convergence as fast as possible. However, an optimization problem has to be solved in a centralized approach and the solution can be applied to a particular graph topology. In addition, some authors demonstrated that predictive consensus algorithms can converge much faster (Aysal, Oreshkin, & Coates, 2009; Kokiopoulou & Frossard, 2009; Oreshkin, Coates, & Rabbat, 2010). In particular, Oreshkin et al. (2010) provide a theoretical demonstration that adding a local prediction component to the update rule can significantly improve the convergence rate of the distributed average

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algorithm. However, the computation of the prediction component needs an overhead in order to evaluate some parameters requiring the spectrum knowledge of the original iteration matrix.

For fast consensus seeking, Jin and Murray (2006) propose protocols that enlarge the algebraic connectivity without physically changing the network topology. Moreover, network communication delays that may occur while exchanging data among multiple agents can degrade the system performances. In this context, Fang et al. (2012) introduce the weighted average prediction into existing consensus protocol to simultaneously impose the robustness to communication delay and the convergence speed achieving the consensus. In addition, the technical note (Tan & Liu, 2013) addresses the consensus problem of discrete-time networked multi-agent systems with network transmission delays, based on a networked predictive control scheme.

In order to investigate consensus protocols with fast asymptotic convergence, we proposed new consensus algorithms in Boschian, Fanti, Mangini, and Ukovich (2011). In particular, we consider the linear system $x = (I - \epsilon L)x$ and, according to the approach of the Point Jacobi and Gauss–Seidel iterative methods to solve large systems of linear equations (Varga, 2000), we presented some consensus algorithms that are based on a positive splitting of matrix $P_\epsilon = (I - \epsilon L)$. However, such protocols did not converge for any network topology.

In this paper, we relax the condition of the positive splitting of matrix P_ϵ and we propose a new class of protocols that are based on a triangular splitting of P_ϵ. The nonnegative matrices theory (Varga, 2000) provides the framework for analyzing the convergence properties of the proposed consensus algorithms. Furthermore, we determine in closed form the protocol that exhibits the following main properties: (i) for each agent network with topology described by a strongly connected graph there exists a triangular splitting that guarantees the convergence at the group decision value; (ii) the consensus algorithm is independent from the value of ϵ; (iii) in each step the agents update the state in a fixed sequence in order to employ the updated state values of the upstream nodes.

The convergence properties are studied and compared with the algorithms proposed in the related literature by means of a number of tests. The results show the good performances of the presented algorithm, even in the cases in which the standard consensus protocol exhibits low convergence speed, i.e., for network topologies described by periodic graphs. In particular, a rigorous theoretical analysis of the proposed protocol convergence is provided for networks with ring topology (a common type of periodic graphs), and compared with the standard algorithm.

The paper is organized as follows. Section 2 describes the problem and Section 3 introduces the new class of consensus algorithms and proves its convergence. Then Section 4 characterizes the triangular splitting that guarantees the convergence. Moreover, Section 5 provides a rigorous comparison for ring topologies between the convergence of the proposed protocol and the standard protocols. Finally, Section 6 summarizes the conclusions.

2. Problem statement

Consider a network of n autonomous agents labeled by an index $i \in V$ with $V = \{1, 2, .., n\}$. Let $x_i \in \Re$ denote the state of the agent i that can represent a physical quantity, such as altitude, position, temperature, voltage, and so on. The interaction topology of the network of agents is represented by a directed graph $G = (V, E)$ where V is the set of nodes and $E \subseteq V \times V$ is the set of edges. Moreover, matrix $A = [a_{ij}]$, with $a_{ij} \in \{0, 1\}$, denotes the adjacency matrix of G, $N_i = \{j \in V : a_{ij} = 1\}$ is the set of neighbors of agent i and $|N_i|$ is its cardinality. More precisely, in the accepted assumption of the related literature, setting $a_{ij} = 1$ denotes the fact that node i can receive information from node j (Olfati-Saber et al., 2007; Olfati-Saber & Murray, 2004; Ren, Beard, & Atkins, 2007). We say that the nodes of a network have reached a consensus if and only if (iff) $x_i = x_j$ for all $i, j \in V$. Furthermore, we define the degree matrix D as the diagonal matrix whose diagonal entries are $D_{ii} = |N_i|$, i.e., the valence of vertex i within the graph. Whenever the agents of a network are all in agreement, the common value of all nodes is called the agreement state and can be expressed as $x^* = \alpha\mathbf{1}$, where $\mathbf{1} = [1, \dots, 1]^T$ and α is a collective decision of the group of the agents. A standard consensus algorithm that solves the agreement problem in a network of agents with discrete-time model is (Olfati-Saber & Murray, 2004):

$$x(k+1) = P_\epsilon\, x(k) \tag{1}$$

where matrix $P_\epsilon = (I - \epsilon L) = [p_{\epsilon ij}]$ is the iteration matrix, ϵ is the step-size parameter, I is the identity matrix and $L = (D - A) = [l_{ij}]$ is the graph Laplacian induced by the graph G. The convergence analysis of the discrete-time consensus algorithm heavily relies on the nonnegative matrix theory (Varga, 2000). Denoting by $\Delta = \max_i \; l_{ii}$ the maximum node out-degree of G, if G is strongly connected, then P_ϵ is a stochastic and irreducible matrix for all $\epsilon \in (0, 1/\Delta)$. Moreover, the decision value is $x^* = \lim_{k\to\infty} x(k) = v w^T x(0)$, where $v = \mathbf{1}$ and $w > 0$ are respectively the right and left eigenvectors of P_ϵ associated with the eigenvalue $\lambda = 1$.

3. The new class of consensus algorithms

This section introduces new consensus algorithms that solve an agreement in networks with fixed or switching topology and zero-communication time delay.

Consider the consensus algorithm (1) and define the following splitting of P_ϵ.

Definition 1. We denote as the triangular splitting of matrix P_ϵ a pair of matrices belonging to the following set: $Q(\epsilon) = \{R \in \Re^{n\times n}, S \in \Re^{n\times n} | R$ is a lower triangular matrix with $r_{ii} \neq 1$ and $r_{ii} \neq 0$ for $i = 1, \dots, n$, S is an upper non-negative triangular matrix and $R + S = P_\epsilon\}$.

The following lemma allows us to prove a property of the triangular splitting of P_ϵ.

Lemma 2. *Let P_ϵ be a stochastic and irreducible matrix and $(R, S) \in Q(\epsilon)$. Then the matrix $(I - R)^{-1}$ exists and is non-negative.*

Proof. By definition it holds $r_{ii} \neq 1$ and $r_{ii} \neq 0$ for $i = 1, \dots, n$. Moreover, since P_ϵ is irreducible and stochastic, then $0 \leq s_{ii} + r_{ii} < 1$. Since by Definition 1 $s_{ii} \geq 0$ and R is a lower triangular matrix, then $r_{ii} < 1$ and $(I - R)$ is non-singular.

Now, in order to prove that $(I - R)^{-1}$ is non-negative, we have to show that $\forall b \geq 0\ \exists x \geq 0$ such that $(I - R)^{-1}x = b$ (Horn & Johnson, 1985).

Consider $b = [b_1 \dots b_n]^T = b_1 e_1 + \dots + b_n e_n$, where e_i for $i = 1, \dots, n$ is the canonical basis of $\Re^n$. Denoting by $x_i = [x_i^1 \ \dots \ x_i^n]^T$ the solution of the iteration $(I - R)x_i = b_i e_i$, we obtain:

$$x_i^j = 0 \quad \text{for } j = 1, \dots, i-1 \text{ and } i = 2, \dots, n \tag{2}$$

$$x_i^j = (1 - r_{ii})^{-1} b_i \quad \text{for } j = i \text{ and } i = 1, \dots, n \tag{3}$$

$$x_i^j = (1 - r_{jj})^{-1} \sum_{k=1}^{j-1} r_{jk} x_i^k \quad \text{for } j > i \text{ and } i = 1, \dots, n. \tag{4}$$

Since $r_{ii} < 1$ for $i = 1, \dots, n$, it is easy to infer by (2)–(4) that $x_i \geq 0$ for $i = 1, \dots, n$ and $x = x_1 e_1 + \dots + x_n e_n \geq 0$, then matrix $(I - R)^{-1}$ is non-negative.

Let us consider the linear system $x = P_\epsilon x$. According to the approach of the Point Jacobi and Gauss–Seidel iterative methods to solve large systems of linear equations (Varga, 2000), we can