Signal Processing ∎ (∎∎∎∎) ∎∎∎–∎∎∎

Contents lists available at ScienceDirect

Signal Processing

journal homepage: www.elsevier.com/locate/sigpro

Fractional order describing functions

J. Tenreiro Machado

Institute of Engineering, Polytechnic of Porto, Department of Electrical Engineering, Porto, Portugal

ARTICLE INFO

Article history:
Received 29 January 2014
Received in revised form
3 April 2014
Accepted 13 May 2014

Keywords:
Fractional calculus
Describing function
Backlash
Nonlinear systems
Limit cycles
Signal propagation

ABSTRACT

This paper addresses limit cycles and signal propagation in dynamical systems with backlash. The study follows the describing function (DF) method for approximate analysis of nonlinearities and generalizes it in the perspective of the fractional calculus. The concept of fractional order describing function (FDF) is illustrated and the results for several numerical experiments are analysed. FDF leads to a novel viewpoint for limit cycle signal propagation as time-space waves within system structure.

1. Introduction

Backlash is a mechanical phenomenon that occurs in gears, because the width of a gear's tooth space exceeds the thickness of an engaging tooth. Backlash is a nonlinearity that degrades control performance, causing vibrations and giving rise to inaccuracies in the system position and velocity. Backlash fingerprints may seem negligible at first sight, namely in smooth signals, but produce artefacts in their time derivatives. The control of systems with backlash has been studied by several researchers [1–12], but well established conclusions and straightforward methods, either for signal processing, or for control implementation, are still lacking.

This paper addresses the analysis of control systems with backlash by means of the describing function (DF) method. The DF method was initially proposed by Nikolaĭ Krylov and Nikolaĭ Bogoliubov in the 1930s [13]. The DF approximates the non-linearity using a function that depends on the amplitude of the input waveform [14,15]. The DF leads to a simple operational procedure for analysing limit cycles, yielding approximate values of its frequency and amplitude [16,17]. The study of backlash in the perspective of DF was investigated, revealing that Fractional Calculus (FC) is a mathematical tool well suited for getting a deeper insight into the phenomenon [18–24].

FC generalizes the operations of integration and differentiation to non-integer orders [25–31]. Researchers verified that fractional order models capture memory effects in cases where classical integer models reveal limitations [32–37]. Having in mind the backlash phenomenon this work embeds the DF and FC and formulates the novel concept of Fractional-order Describing Functions (FDFs). In other words, this paper presents the *conjecture* of having FDF similar to what is presently accepted as Fractional-order Transfer Function (FTF).

The paper is organized as follows. Section 2 presents the fundamentals of DF and the model for static backlash. Section 3 proposes the concept of FDF and develops an operational method for its analytical implementation. Section 4 addresses the outcome of the FDF proposed concept in what concerns the signal propagation in chains. Finally, Section 5 draws the main conclusions and addresses future developments.

2. Describing function and backlash

Consider a control system ($\mathcal{CS}$) with a forward loop having a nonlinear element followed by the TF of a linear

E-mail address: jtm@isep.ipp.pt

http://dx.doi.org/10.1016/j.sigpro.2014.05.012
0165-1684/

system $G(s)$, and unit feedback. Suppose that the input signal, $x(t)$, that feeds the nonlinear element is

$$x(t) = X \cos(\omega t) \tag{1}$$

where X and ω denote the amplitude and the angular frequency, respectively.

The output signal of the nonlinear element, $y(t)$, is not sinusoidal. In general, the output is periodic, with period identical to the input signal, and includes higher harmonics besides the fundamental component. If the nonlinearity is symmetrical with respect to the variation around zero, the Fourier series results

$$y(t) = \sum_{k=1}^{\infty} Y_k \cos(k\omega t + \varphi_k) \tag{2}$$

where Y_k and φ_k, $k = 1, \ldots, \infty$, are the amplitude and the phase shift of the k-th harmonic, respectively.

The DF method assumes that only the first harmonic of the output $Y_1 \cos(\omega t + \varphi_1)$ is relevant. This is usually valid, not only because the higher harmonics present in $y(t)$ have a small amplitude, but also because $G(j\omega)$ is often a 'low-pass filter' attenuating even further the harmonics [14,15,38].

The DF of a nonlinear element, $N(X, \omega)$, is defined as the complex ratio of the first harmonic components

$$N(X, \omega) = \frac{Y_1 e^{j\varphi_1}}{X} \tag{3}$$

where $j = \sqrt{-1}$.

When the nonlinear element involves energy (storage or dissipation) the DF depends on both the amplitude and the frequency, $N(X, \omega)$. Otherwise, if the nonlinearity depends only on the amplitude, then the DF is of the form $N(X)$. In general, the product of two DF may be misleading since we are working with nonlinear systems. Therefore, two nonlinear elements in *series*, with DF denoted by N_1 and N_2, are not modelled by the product of the two individual DF, being necessary to calculate the corresponding DF N_{12} (i.e., $N_1 \cdot N_2 \neq N_{12}$). Two nonlinear elements in *parallel* have a DF that corresponds to the sum of the two individual DF (i.e., $N_1 + N_2 = N_{12}$). For example, it is well known that the DF of the dead-zone and saturation blocks in parallel and having identical parameters corresponds to a linear gain. The characteristics of *series* and *parallel* associations of DF are rarely mentioned, but their properties are useful in the sequel.

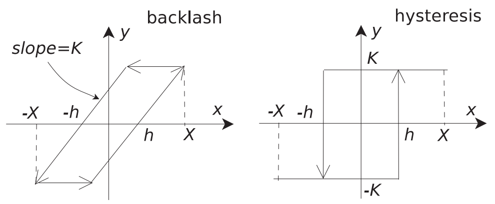

Fig. 1. Block diagrams of the backlash and hysteresis nonlinearities.

The DF is used for the approximate stability analysis of the closed-loop nonlinear control system. For the $\mathcal{CS}$ mentioned previously, if the high order harmonics are sufficiently attenuated, then N can be interpreted as a complex gain and the closed-loop characteristic equation in the frequency domain yields

$$N(X)G(j\omega) + 1 = 0 \tag{4}$$

If Eq. (4) has one or more solutions, then a *limit cycle* is predicted with frequency ω_0 and amplitude X_0 such that $N(X_0)G(j\omega_0) + 1 = 0$. It must be noted that (4) is valid only if $\mathcal{CS}$ is in a steady-state. Therefore, the DF analysis predicts only the presence, or the absence, of a limit cycle and cannot be applied to estimate other types of signals and time responses.

Several DF analytical expressions of standard nonlinear elements can be found in [39,40]. The static backlash (or, friction-controlled backlash) and the hysteresis (or, rectangular hysteresis) represented in Fig. 1 have DF given by the expressions

$$N(X) = \frac{K}{2} - \frac{K}{\pi}\left[\sin^{-1}\left(\frac{2h}{X} - 1\right) + 2\left(\frac{2h}{X} - 1\right)\sqrt{\frac{h}{X}\left(1 - \frac{h}{X}\right)}\right]$$
$$-j\frac{4Kh(X-h)}{\pi X^2}, \quad X > h \tag{5}$$

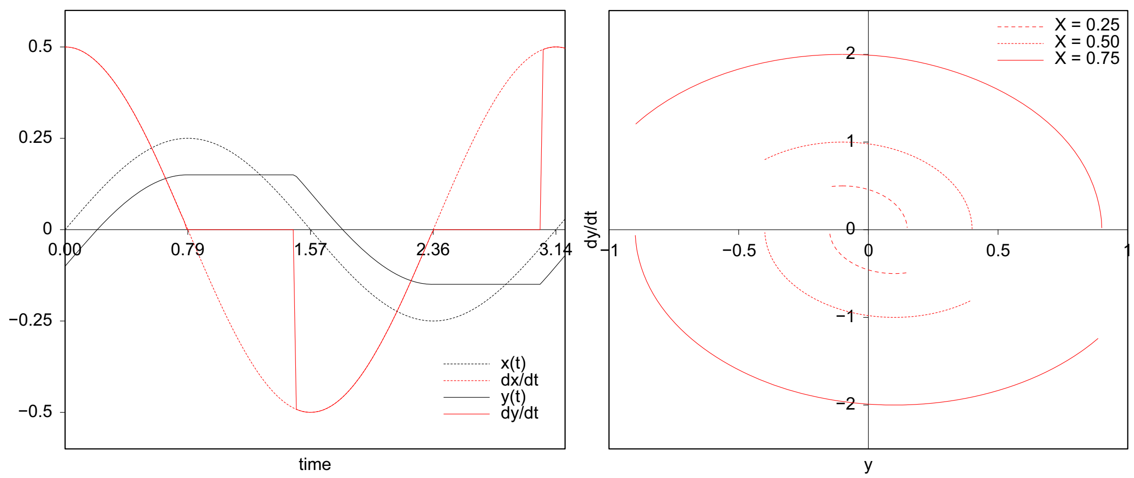

Fig. 2. Backlash with $\{K, h\} = \{1, 0.1\}$: Time response for $X = 0.25$ and output phase plane $(y, \dot{y})$ for $X = \{0.25, 0.5, 1.0\}$.

Please cite this article as: J. Tenreiro Machado, Fractional order describing functions, Signal Processing (2014), http://dx.doi.org/10.1016/j.sigpro.2014.05.012