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New conditions for uniformly recovering sparse signals via orthogonal matching pursuit

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ABSTRACT

Recently, lots of work has been done on conditions of guaranteeing sparse signal recovery using orthogonal matching pursuit (OMP). However, none of the existing conditions is both necessary and sufficient in terms of the so-called restricted isometric property, coherence, cumulative coherence (Babel function), or other verifiable quantities in the literature. Motivated by this observation, we propose a new measure of a matrix, named as union cumulative coherence, and present both sufficient and necessary conditions under which the OMP algorithm can uniformly recover sparse signals for all sensing matrices. The proposed condition guarantees a uniform recovery of sparse signals using OMP, and reveals the capability of OMP in sparse recovery. We demonstrate by examples that the proposed condition can be used to more effectively determine the recoverable sparse signals via OMP than the conditions existing in the literature. Furthermore, sparse recovery from noisy measurements is also considered in terms of the proposed union cumulative coherence.

1. Introduction

Orthogonal matching pursuit (OMP) as a canonical greedy algorithm for signal approximation is frequently used for sparse recovery in compressive sensing [1]. In comparison with other sparse recovery algorithms, the major advantage of OMP is its simplicity and easy implementation [2,3]. Besides, OMP has very competitive performance in recovering sparse signals [4], and has been investigated extensively. Several variants of OMP, such as regularized orthogonal matching pursuit (ROMP), compressive sampling matching pursuit (CoSaMP), and subspace pursuit (SP), have been proposed to further improve its performance [5–8].

A key theoretical issue of OMP is to determine the condition on which OMP can uniformly recover sparse signals. Recently, many conditions have been derived in terms of the restricted isometric property (RIP) [2,9–12]. For example, it was proven that OMP can recover all k-sparse signals if $\delta_{k+1} < 1/(3\sqrt{k})$ [2], $\delta_k < \sqrt{k-1}/(\sqrt{k-1}+k)$ [9], $\delta_{k+1} < 1/\sqrt{2k}$ [10] or $\delta_{k+1} < 1/(\sqrt{k}+1)$ [11,12], where δ_k is the restricted isometric constant (RIC) of order k. In addition to RIC, coherence and cumulative coherence were also used to build the condition for sparse recovery [1,3,4]. Particularly, Tropp has investigated the convergence of OMP using the cumulative coherence $\mu_1(k)$, and shown that when $\mu_1(k-1)+\mu_1(k) < 1$, OMP can exactly recover all k -sparse signals [3]. Tropp and Gilbert also used the coherence of a random matrix to show the effectiveness of signal recovery

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from random measurements via OMP [4]. Following the above-mentioned work, in this paper we aim to further explore the condition of sparse recovery via OMP.

Although all the conditions mentioned above attempt to better formulate the requirement for uniformly recovering sparse signals via OMP, they are only sufficient, and none of them is necessary. It means that according to these conditions, only some k-sparse signals can be identified to be uniformly recoverable by OMP. Our objective in this paper is to present a sufficient and necessary condition for uniformly recovering sparse signals via OMP in terms of a new measurement called union cumulative coherence. Specifically, by defining a union cumulative coherence $\tilde{\mu}_1(k)$ of a matrix or dictionary of order k, we prove that $\tilde{\mu}_1(k) < 1$ is both sufficient and necessary for all sensing matrices to exactly recover all k-sparse signals using OMP. The distinct feature of the proposed condition is that it adopts a verifiable measure, similar to the RIP constant, coherence and cumulative coherence, to determine exactly whether or not sparse signals can be uniformly recovered via OMP for any sensing matrix. We demonstrate through some examples that the new condition gives a much larger sparsity bound of recoverable signals than the RIC based conditions as mentioned above. Here, the sparsity bound means the largest number of nonzero components of a sparse signal. It is noted that, although a precise condition of sparse recovery via OMP was presented in [3], it is not very useful since there is not any technique for checking when the condition holds as the author pointed out. On account of this, Tropp introduced cumulative coherence μ_1 to give a sufficient condition as mentioned above. Compared with the μ_1 based condition, our new condition is both sufficient and necessary for uniform recovery of sparse signals via OMP uniformly for all sensing matrices. As shown by examples, there exist matrices that do not satisfy the cumulative coherence condition but still satisfy our condition. Also, the union cumulative coherence can be used to analyze the noisy recovery of sparse signals, and a condition of reconstructing sparse signals from noisy measurements is derived with the new measure.

The idea behind the union cumulative coherence arises naturally from the manner of identifying the indexes in the support of a sparse signal in the OMP algorithm. The proof of the new condition is straightforward. Unlike the coherence and cumulative coherence, the union cumulative coherence of a dictionary is the maximum of the sum of two interrelated cumulative coherences: one is the cumulative coherence between one atom ϕ and all the elements in a collection $\mathcal{T}$ of atoms different from ϕ, and the other is that between one atom and the others in $\mathcal{T}$. They are connected to each other by the common collection $\mathcal{T}$. This is the fundamental difference between the union cumulative coherence and cumulative coherence. By such a refined measure, we can develop a precise condition of uniformly recovering sparse signals via OMP. It is worth mentioning that, although most conditions of sparse recovery in the literature were based on RIC, we can demonstrate by examples that the RIC constant cannot be used to formulate a precise measure that guarantees a uniform recovery of sparse signals by OMP.

The rest of the paper is organized as follows. The OMP algorithm is described briefly in Section 2. In Section 3, a sufficient and necessary condition for uniform recovery of sparse signals is presented. The sparse recovery with noisy measurements is discussed in terms of the union cumulative coherence in Section 4. Section 5 concludes the paper.

Notations: $\operatorname{supp} \boldsymbol{\alpha} = \{i: \alpha_i \neq 0\}$ is the support of a vector $\boldsymbol{\alpha} = (\alpha_i)$. For $\Lambda \subseteq \{1, \ldots, M\}$, $|\Lambda|$ denotes the cardinality of Λ, and $\Lambda^c = \{1, \ldots, M\} \backslash \Lambda$ is the set of all elements of $\{1, \ldots, M\}$ not contained in Λ. $\boldsymbol{\Phi} = (\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_M)$ is a matrix consisting of the column vectors $\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_M \in \mathbb{R}^N$. $\boldsymbol{\Phi}_\Lambda$ denotes a submatrix of $\boldsymbol{\Phi}$ that only contains columns indexed by Λ. $\boldsymbol{\Phi}^\dagger$ is the Moore–Penrose pseudoinverse of $\boldsymbol{\Phi}$, and $\boldsymbol{\Phi}^T$ the transpose of $\boldsymbol{\Phi}$. For a square matrix $\boldsymbol{A} = (a_{ij})$, det $(\boldsymbol{A})$ denotes its determinant, and diag $(a_1, \ldots, a_n)$ represents a diagonal matrix with $a_{ij} = 0$ for $1 \le i \neq j \le n$ and $a_{ii} = a_i$ for $1 \le i \le n$. The inner product of vectors $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ is written as $\langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle$.

2. Recovery of sparse signals using OMP

Given a vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_M) \in \mathbb{R}^M$, $\boldsymbol{\alpha}$ is called k-sparse if $|\operatorname{supp} \boldsymbol{\alpha}| \le k$. The recovery of sparse signals, or simply sparse recovery, is to reconstruct a sparse vector $\boldsymbol{\alpha}$ from the noiseless measurement $\boldsymbol{y} = \boldsymbol{\Phi}\boldsymbol{\alpha}$ or noisy measurement $\boldsymbol{y} = \boldsymbol{\Phi}\boldsymbol{\alpha} + \boldsymbol{\varepsilon}$, where $\boldsymbol{\Phi}$ is an $N \times M$ matrix with $N < M$ and $\boldsymbol{\varepsilon}$ an independent noise. The basic idea of OMP is to sequentially identify the columns of Φ participating in the vector $\boldsymbol{y}$. In principle, the algorithm selects the column of Φ which is most strongly correlated with the remaining part of $\boldsymbol{y}$, and subtracts off the contribution of the chosen column from the remaining part of $\boldsymbol{y}$ such that the residual is orthogonal to all vectors chosen, and then iterates on the residual. For a k-sparse signal $\boldsymbol{\alpha}$, if all the indexes of its nonzero entries are correctly determined after at most k iterations, OMP can exactly recover $\boldsymbol{\alpha}$ from its noiseless measurements.

Algorithm 1. OMP algorithm.

Input: measurement matrix $\boldsymbol{\Phi} = (\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_M)$, measurement vector $\boldsymbol{y}$, sparsity level k
Output: sparse signal $\boldsymbol{\alpha}$ and its support $\Lambda \subset \{1, \ldots, M\}$
Initialize: $\Lambda = \varnothing$, $\boldsymbol{y}_r = \boldsymbol{y}$
Repeat : do steps (1)–(2) k times
(1) Find the index i such that $i = \arg \max_{m \in \{1,\ldots,M\}} |\langle \boldsymbol{y}_r, \boldsymbol{\phi}_m \rangle|$ (if the maximum occurs for multiple indices, choose one randomly)
(2) Update the index set and the residual: $\Lambda = \Lambda \cup \{i\}$,
$\boldsymbol{\alpha} = \arg \min_{\boldsymbol{\beta}} \| \boldsymbol{y}_r - \boldsymbol{\Phi}_\Lambda \boldsymbol{\beta} \|, \quad \boldsymbol{y}_r = \boldsymbol{y}_r - \boldsymbol{\Phi}_\Lambda \boldsymbol{\alpha}$
(3) Let $\tilde{\boldsymbol{\alpha}} = \boldsymbol{\Phi}_\Lambda^\dagger \boldsymbol{y}$, $\boldsymbol{\alpha}(i) = \tilde{\boldsymbol{\alpha}}(i)$ for $i \in \Lambda$ and 0 for $i \notin \Lambda$, $1 \le i \le M$.

Since OMP is a stepwise forward selection algorithm, the analysis of whether it correctly recovers sparse signals can be concentrated on its each iteration. For any k-sparse signal expressed as a linear combination of the k columns of a matrix $\boldsymbol{\Phi}$, we only need to inspect the relation between arbitrary k columns and any other one at each iteration. It is obvious that if there is a condition that can guarantee that OMP is able to correctly identify one index in the support of any k-sparse vector at each iteration, then it will ensure that OMP can uniformly recover k-sparse